

Borsuk's Theorem

MICHAEL TAYLOR

1. Introduction

The following result on antipodal-preserving maps on the n -sphere S^n is known as Borsuk's theorem.

Theorem 1.1. *If $f : S^n \rightarrow S^n$ is a continuous map satisfying*

$$(1.1) \quad f(-x) = -f(x), \quad \forall x \in S^n,$$

then $\text{Deg } f$ is odd.

This has a number of interesting consequences, including the following, called the Borsuk-Ulam theorem.

Corollary 1.2. *There exists no continuous map $f : S^n \rightarrow S^{n-1}$ satisfying (1.1).*

Proof. The degree of a continuous map $f : S^n \rightarrow S^n$ with range in S^{n-1} must be zero, which is not odd.

Corollary 1.3. *If $h : S^n \rightarrow \mathbb{R}^n$ is continuous and satisfies $h(-x) = -h(x)$ for all $x \in S^n$, then there exists $x \in S^n$ such that $h(x) = 0$.*

Proof. If not, $f(x) = h(x)/|h(x)|$ satisfies the conditions forbidden by Corollary 1.2.

Corollary 1.4. *If $g : S^n \rightarrow \mathbb{R}^n$ is continuous, then there are two antipodal points on S^n with the same image.*

Proof. If not, $h(x) = g(x) - g(-x)$ must map S^n to $\mathbb{R}^n \setminus 0$, and satisfy $h(-x) = -h(x)$. Corollary 1.3 forbids this.

A number of proofs of Theorem 1.1 have been produced, bringing to bear a variety of tools, such as intersection theory ([GP], Chapter 2), and homology with coefficients in $\mathbb{Z}/(2)$ ([H], Chapter 2). A particularly intriguing approach appears in Chapter 2, §11 of [CH]. However, the proof proposed there seems to have a gap. We discuss this in §2.

Another consequence of Borsuk's theorem, known as the ham sandwich theorem, will be discussed in §3.

In preparation for §2, we give a brief sketch of the notion of degree, using the approach taken in Chapter 1, §§19–20 of [T1], and in Chapter 5, §3 of [T2]. Say X and Y are smooth, compact, oriented n -dimensional manifolds, and Y is connected. If

$$(1.2) \quad f : X \longrightarrow Y \text{ is smooth,}$$

we take α to be an n -form on Y , satisfying $\int_Y \alpha = 1$, and set

$$(1.3) \quad \text{Deg } f = \int_X f^* \alpha.$$

If also $\alpha_1 \in \Lambda^n Y$ integrates to 1, then Proposition 19.5 in Chapter 1 of [T1] implies that $\alpha - \alpha_1 = d\beta$ for some $\beta \in \Lambda^{n-1} Y$, hence

$$(1.4) \quad \int_X f^* \alpha - \int_X f^* \alpha_1 = \int_X f^* d\beta = \int_X df^* \beta = 0,$$

the last identity by Stokes' theorem, so $\text{Deg } f$ is well defined, for smooth f . Homotopy invariance (Proposition 19.7 of [T1]) allows for a well defined extension to continuous f . Going back to smooth f , it is seen in Proposition 19.8 of [T1] that, if $y_0 \in Y$ is a regular value of f (these exist, by Sard's theorem), then

$$(1.5) \quad \text{Deg } f = \sum \left\{ \text{sgn det } Df(x_j) : f(x_j) = y_0 \right\},$$

hence $\text{Deg } f$ is an integer. Proposition 20.2 of [T1] says the following.

Proposition 1.5. *Let $\bar{\Omega}$ be a smoothly bounded region in \mathbb{R}^{n+1} . Let V be a smooth vector field on $\bar{\Omega}$, with a finite set of zeroes p_j , all in the interior Ω . Define $F : \partial\Omega \rightarrow S^n$ by $F(x) = V(x)/|V(x)|$. Then*

$$(1.6) \quad \text{Index } V = \text{Deg } F.$$

As seen in (20.4) of [T1], if all these zeroes $\{p_j\}$ of V are non-degenerate, then

$$(1.7) \quad \text{Index } V = \sum_j \text{sgn det } DV(p_j).$$

The approach to degree taken in [CH], adapted from [N], basically takes (1.6)–(1.7) as a definition of $\text{Deg } F$ (hence requiring an argument that $\text{Deg } F$ is well defined, somewhat different from the use of (1.3)–(1.4) above).

We proceed as follows. In §2 we show how Theorem 1.1 follows from a result given as Lemma 2.1, describe the argument in [CH] to establish this lemma, and point out a flaw in this argument. We then state a conjectured approach to a proof of the lemma. In §2A we present an alternative conjectural approach. In §3 we state a version of the ham sandwich theorem and derive it from Corollary 1.3. In §4 we discuss a (well known) elementary proof of Corollary 1.2 (hence of Corollary 1.3) in the special case of a map $f : S^2 \rightarrow S^1$.

2. Approach to the proof of Theorem 1.1

The approach to Theorem 1.1 taken in [CH] brings in the following.

Lemma 2.1. *Let $f : S^n \rightarrow S^n$ be continuous and satisfy (1.1). Pick $\varepsilon > 0$. Then there exists a smooth map $V : \overline{B}^{n+1} \rightarrow \mathbb{R}^{n+1}$ such that*

$$(2.1) \quad V(-x) = -V(x), \quad \forall x \in \overline{B}^{n+1},$$

$$(2.2) \quad \text{each zero of } V \text{ is nondegenerate,}$$

and

$$(2.3) \quad \frac{V}{|V|} \Big|_{S^n} = F : S^n \rightarrow S^n \text{ is uniformly within } \varepsilon \text{ of } f.$$

Here $\overline{B}^{n+1} = \{x \in \mathbb{R}^{n+1} : |x| \leq 1\}$. To deduce Theorem 1.1 from this lemma, use (1.6)–(1.7) and note that if (2.1) holds, then nonzero critical points of V occur in pairs $\pm p_j$, hence

$$(2.4) \quad \begin{aligned} \text{Deg } F &= \text{sgn det } DV(0) \pmod{2} \\ &= 1 \pmod{2}, \end{aligned}$$

hence

$$(2.5) \quad \text{Deg } f = \text{Deg } F = 1 \pmod{2},$$

as desired.

We turn to steps proposed in [CH] to establish Lemma 2.1. First, we can mollify f , still satisfying (2.1), and then extend to a smooth function $V_1 : \overline{B}^{n+1} \rightarrow \mathbb{R}^{n+1}$, satisfying (2.1) and (2.3), but possibly not satisfying (2.2). Next, change V_1 slightly, to V_2 , satisfying (2.2), and also (2.3). It is proposed in [CH] that one can obtain the desired function V as

$$(2.6) \quad V(x) = \frac{1}{2} [V_2(x) - V_2(-x)].$$

Let's work this out in more detail. The standard way to pass from V_1 to V_2 is to note that, by Sard's theorem, the set of critical values of V_1 is a subset of \mathbb{R}^{n+1} of measure 0. Pick a small $w_\varepsilon \in \mathbb{R}^{n+1}$, a regular value of V_1 , and set

$$(2.7) \quad V_2(x) = V_1(x) - w_\varepsilon,$$

so the set of zeroes of V_2 is $\{x \in B^{n+1} : V_1(x) = w_\varepsilon\}$. Of course, (2.1) fails for V_2 . However, applying (2.6) gives

$$\begin{aligned}
 (2.8) \quad V(x) &= \frac{1}{2} [V_1(x) - w_\varepsilon - V_1(-x) + w_\varepsilon] \\
 &= \frac{1}{2} (V_1(x) - V_1(-x)) \\
 &= V_1(x),
 \end{aligned}$$

and we're back to square 1. We have failed to achieve (2.2).

So the argument sketched above fails to prove Lemma 2.1. However, this lemma looks too good to be false. Here's a proposed fix.

Conjecture. Assume $V_1 : \overline{B}^{n+1} \rightarrow \mathbb{R}^{n+1}$ is smooth, $V_1|_{S^n} = f : S^n \rightarrow S^n$, and $V_1(-x) = -V_1(x)$ for all $x \in \overline{B}^{n+1}$. Then, for almost all sufficiently small $A \in M(n+1, \mathbb{R})$,

$$(2.9) \quad V(x) = V_1(x) + Ax$$

satisfies (2.2), hence (2.1)–(2.3).

2A. Another approach to Lemma 2.1

Take $f : S^n \rightarrow S^n$ to be smooth and satisfy (1.1). Define $V_1 : \overline{B}^{n+1} \rightarrow \overline{B}^{n+1}$ by

$$(2A.1) \quad V_1(x) = |x|^2 f\left(\frac{x}{|x|}\right).$$

Then V_1 is C^1 . It has just one zero, at $x = 0$, highly degenerate. Take $\alpha > 0$ small, and consider

$$(2A.2) \quad V_R(x) = V_1(x) - \alpha R x, \quad R \in SO(n+1).$$

Then $x = 0$ is a nondegenerate zero of V_R . Note that

$$(2A.3) \quad |V_1(x)| = |x|^2, \quad |\alpha R x| = \alpha |x|,$$

so all the other zeroes of V_R lie on the sphere $\{x \in \mathbb{R}^{n+1} : |x| = \alpha\}$. We make the following

Conjecture 2A.1. *For almost all $R \in SO(n+1)$, V_R has only a finite number of zeroes in $\{x : |x| = \alpha\}$.*

Given this conjecture, we can prove Theorem 1.1 as follows. From (1.1) and (2A.2) we have

$$(2A.4) \quad V_R(-x) = -V_R(x),$$

so the zeroes of V_R on $\{x : |x| = \alpha\}$ occur in pairs $\pm p_j$. We can hence take smoothly bounded regions $\overline{\Omega}_\pm \subset B^{n+1}$, which are mutually disjoint and satisfy

$$(2A.5) \quad \Omega_- = -\Omega_+,$$

and all the zeroes of V_R in $\{x : |x| = \alpha\}$ are contained in $\Omega_+ \cup \Omega_-$. It follows from (2A.4)–(2A.5) and Proposition 1.5 that

$$(2A.6) \quad \text{Index } V_R|_{\Omega_+} = \text{Index } V_R|_{\Omega_-},$$

so $\text{Index } V_R|_{\Omega_+ \cup \Omega_-}$ is an even integer. Then

$$(2A.7) \quad \begin{aligned} \text{Index } V_R &= \text{sgn det } DV_R(0) + \text{Index } V_R|_{\Omega_+ \cup \Omega_-} \\ &= 1 \pmod{2}, \end{aligned}$$

and we have Theorem 1.1.

Turning to Conjecture 2A.1, we note that

$$(2A.8) \quad \alpha^{-2} V_R(\alpha x) = f(x) - R x, \quad \text{for } |x| = 1.$$

Hence Conjecture 2A.1 is a consequence of the following.

Conjecture 2A.2. *Let $f : S^n \rightarrow S^n$ be smooth. Then, for almost all $R \in SO(n+1)$,*

$$(2A.9) \quad f_R(x) = f(x) - Rx$$

has only a finite number of zeros on S^n .

Replacing R by R^{-1} , we get yet another formulation:

Conjecture 2A.3. *Let $f : S^n \rightarrow S^n$ be smooth. Then, for almost all $R \in SO(n+1)$,*

$$(2A.10) \quad \varphi_R = R \circ f : S^n \longrightarrow S^n$$

has only a finite number of fixed points on S^n .

Tiptoeing around these conjectures, we note the following related result.

Proposition 2A.4. *Let G be a compact Lie group, and let $f : G \rightarrow G$ be smooth. Then, for almost all $g \in G$, the equation*

$$(2A.11) \quad f(x) = gx$$

has at most a finite number of solutions $x \in G$.

Proof. The equation (2A.11) is equivalent to

$$(2A.12) \quad f(x)x^{-1} = g,$$

so the result follows from the fact that

$$(2A.13) \quad \psi : G \longrightarrow G, \quad \psi(x) = f(x)x^{-1}$$

has the property that almost every $g \in G$ is a regular value of ψ (by Sard's theorem).

Taking $n = 3$ in Conjecture 2A.2, and using $S^3 \approx Sp(1) \subset SO(4)$, we see that Conjecture 2A.2 holds if we replace $SO(4)$ by $Sp(1)$. This is enough to prove Theorem 1.1 in the case $n = 3$.

3. The ham sandwich theorem

In this section we establish a generalization of a result fancifully known as the ham sandwich theorem. To state it, we bring in some terminology. Let N be a unit vector in \mathbb{R}^n , and for $t \in \mathbb{R}$ define the hyperplane

$$(3.1) \quad H(N, t) = \{x \in \mathbb{R}^n : x \cdot N = t\},$$

which bounds two half spaces,

$$(3.2) \quad \mathcal{H}^\pm(N, t) = \{x \in \mathbb{R}^n : \pm(x \cdot N - t) > 0\}.$$

Let μ_1, \dots, μ_n be probability measures on \mathbb{R}^n . Assume that, whenever $H \subset \mathbb{R}^n$ is a hyperplane, $\mu_j(H) = 0$ for all $j \in \{1, \dots, n\}$. Here is the result.

Theorem 3.1. *There is a hyperplane $H = H(N, t) \subset \mathbb{R}^n$ such that*

$$(3.3) \quad \mu_j(\mathcal{H}^\pm(N, t)) = \frac{1}{2}, \quad \forall j \in \{1, \dots, n\}.$$

Proof. It is convenient first to introduce the probability measures

$$(3.4) \quad \mu_j^\varepsilon = (1 - \varepsilon)\mu_j + \varepsilon\nu,$$

where ν is a probability measure of the form $\nu = \varphi\lambda$, λ denoting Lebesgue measure on \mathbb{R}^n and $\varphi : \mathbb{R}^n \rightarrow (0, \infty)$ a continuous function integrating to 1. For $\varepsilon \in (0, 1)$, define

$$(3.5) \quad f_{j\varepsilon} : S^{n-1} \longrightarrow \mathbb{R}, \quad j \in \{1, \dots, n\},$$

so that $f_{j\varepsilon}(N)$ is the unique $t_{j\varepsilon} \in \mathbb{R}$ such that

$$(3.6) \quad \mu_j^\varepsilon(\{x \in \mathbb{R}^n : x \cdot N > t_{j\varepsilon}\}) = \frac{1}{2}.$$

Note that $f_{j\varepsilon}$ is continuous and

$$(3.7) \quad f_{j\varepsilon}(-N) = -f_{j\varepsilon}(N), \quad \forall N \in S^{n-1}.$$

We claim that there exists $N_\varepsilon \in S^{n-1}$ such that

$$(3.8) \quad f_{1\varepsilon}(N_\varepsilon) = \dots = f_{n\varepsilon}(N_\varepsilon).$$

Given that this is the case, we have the following variant of (3.3):

$$(3.9) \quad \mu_j^\varepsilon(\mathcal{H}^\pm(N_\varepsilon, t_\varepsilon)) = \frac{1}{2}, \quad t_\varepsilon = f_{1\varepsilon}(N_\varepsilon) = \cdots = f_{n\varepsilon}(N_\varepsilon).$$

We can also put a uniform bound on $|t_\varepsilon|$. Taking a sequence $\varepsilon_\ell \rightarrow 0$, we can pass to a subsequence such that $N_{\varepsilon_\ell} \rightarrow N_0 \in S^{n-1}$, $t_{\varepsilon_\ell} \rightarrow t_0 \in \mathbb{R}$, and then (3.3) holds for $\mathcal{H}^\pm(N_0, t_0)$.

It remains to establish the claim (3.8). For this, define

$$(3.10) \quad \begin{aligned} F^\varepsilon : S^{n-1} &\longrightarrow \mathbb{R}^{n-1}, \\ F^\varepsilon(N) &= (f_{1\varepsilon}(N) - f_{n\varepsilon}(N), \dots, f_{n-1,\varepsilon}(N) - f_{n\varepsilon}(N)). \end{aligned}$$

Then F^ε is continuous and

$$(3.11) \quad F^\varepsilon(-N) = -F^\varepsilon(N), \quad \forall N \in S^{n-1},$$

so Corollary 1.3 implies $F^\varepsilon(N_\varepsilon) = 0$ for some $N_\varepsilon \in S^{n-1}$. Then N_ε satisfies (3.8), and we are done.

4. Elementary case of Corollary 1.2

Here we treat the following relatively elementary case of Corollary 1.2, which suffices for the proof of the ham sandwich theorem on \mathbb{R}^3 :

Proposition 4.1. *There exists no continuous map $f : S^2 \rightarrow S^1$ satisfying*

$$(4.1) \quad f(-x) = -f(x), \quad \forall x \in S^2.$$

Proof. Since S^2 is simply connected, each continuous map $f : S^2 \rightarrow S^1$ has a lift,

$$(4.2) \quad \varphi : S^2 \longrightarrow \mathbb{R},$$

such that

$$(4.3) \quad f(x) = (\cos \varphi(x), \sin \varphi(x), 0).$$

If (4.1) holds, then

$$(4.4) \quad (\cos \varphi(-x), \sin \varphi(-x)) = -(\cos \varphi(x), \sin \varphi(x)),$$

hence, for all $x \in S^2$,

$$(4.5) \quad \varphi(x) - \varphi(-x) \in \{(2k + 1)\pi : k \in \mathbb{Z}\}.$$

Since the right side of (4.5) is discrete and φ is continuous, the left side must be constant, so there exists $k \in \mathbb{Z}$ such that

$$(4.6) \quad \varphi(x) - \varphi(-x) = (2k + 1)\pi, \quad \forall x \in S^2.$$

But then we can substitute $-x$ for x in (4.6), obtaining

$$(4.7) \quad \varphi(-x) - \varphi(x) = (2k + 1)\pi,$$

a contradiction.

References

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