Generalized Eigenspace Decomposition

Michael Taylor

Let V be a complex vector space, $\dim V = n < \infty$, and take $T \in \mathcal{L}(V)$. The roots $\{\lambda_j\}$ of $\det(\lambda I - T)$ are the eigenvalues of T. We set

$$\mathcal{GE}(T,\lambda_j) = \{v \in V : (T-\lambda_j)^k v = 0 \text{ for some } k\}. \tag{1}$$

We aim to show that

$$V = \bigoplus_j \mathcal{GE}(T,\lambda_j). \tag{2}$$

We follow the elegant argument in Lecture 9 of [G], with a few tweaks.

Start with generalities. For $A \in \mathcal{L}(V)$, set

$$\mathcal{N}^{\#}(A) = \bigcup_k \mathcal{N}(A^k), \quad \mathcal{R}^{\#}(A) = \bigcap_k \mathcal{R}(A^k), \tag{3}$$

where $\mathcal{N}(A)$ is the null space of A and $\mathcal{R}(A)$ is its range. We have stabilization: for some m,

$$\mathcal{N}^{\#}(A) = \mathcal{N}(A^m) = \mathcal{N}(S), \quad \mathcal{R}^{\#}(A) = \mathcal{R}(A^m) = \mathcal{R}(S), \tag{4}$$

where $S = A^m$. Note that

$$A, S : \mathcal{R}(S) \longrightarrow \mathcal{R}(S) \ \text{ are onto, hence isomorphisms.} \tag{5}$$

Lemma 1. *We have*

$$V = \mathcal{N}(S) \oplus \mathcal{R}(S) = \mathcal{N}^{\#}(A) \oplus \mathcal{R}^{\#}(A). \tag{6}$$

Proof. The rank-nullity theorem (aka, the fundamental theorem of linear algebra) gives

$$\dim V = \dim \mathcal{N}(S) + \dim \mathcal{R}(S), \tag{7}$$

so it suffices to note that, when (5) holds,

$$\mathcal{N}(S) \cap \mathcal{R}(S) = \{0\}. \tag{8}$$

We apply this to

$$A = T - \lambda_j I, \tag{9}$$

where we pick an eigenvalue λ_j of T. So (6) says

$$V = \mathcal{GE}(T,\lambda_j) \oplus \mathcal{R}^{\#}(T-\lambda_j I), \tag{10}$$

and (5) implies that

$$T - \lambda_j I : \mathcal{R}^{\#}(T-\lambda_j I) \xrightarrow{\approx} \mathcal{R}^{\#}(T-\lambda_j I), \tag{11}$$

and consequently

$$T : \mathcal{R}^{\#}(T-\lambda_j I) \longrightarrow \mathcal{R}^{\#}(T-\lambda_j I). \tag{12}$$

We are ready for the main result:

Theorem 2. *The direct decomposition (2) holds.*

Proof. Use induction on $\dim V$. If λ_j is an eigenvalue, then $\dim \mathcal{GE}(T,\lambda_j) \ge 1$, so (10)–(12) hold, with $\dim \mathcal{R}^{\#}(T-\lambda_j I) < \dim V$. Inductively,

$$\mathcal{R}^{\#}(T-\lambda_j I) = \bigoplus_{k\neq j} \mathcal{GE}(T,\lambda_k), \tag{13}$$

and we are done.

Remark. We have

$$\det(\lambda I - T) = \prod_j \det(\lambda I - T_j), \quad T_j = T\big|_{\mathcal{GE}(T,\lambda_j)} \in \mathcal{L}(\mathcal{GE}(T,\lambda_j)), \tag{14}$$

and each $T_j = \lambda_j I + N_j$, with N_j nilpotent, so $\mathcal{GE}(T,\lambda_j)$ has a basis in which the matrix of N_j is strictly upper triangular. Hence

$$\det(\lambda I - T_j) = (\lambda - \lambda_j)^{d_j}, \quad d_j = \dim \mathcal{GE}(T,\lambda_j). \tag{15}$$

Remark 2. Generally, if N is nilpotent on a vector space of dimension d, then $N^d = 0$. In particular, $(T-\lambda_j I)^{d_j}|_{\mathcal{GE}(T,\lambda_j)} = 0$. Hence if $C_T(\lambda) = \det(\lambda I - T)$ denotes the characteristic polynomial of T, we have from (14)–(15), together with (2), that

$$C_T(T) = \prod_j (T-\lambda_j I)^{d_j} = 0, \tag{16}$$

which is the Cayley-Hamilton theorem.

References

[G] C. Grant, Theory of Ordinary Differential Equations, Lecture Notes for Math 634, Brigham Young Univ., available at http://www.math.byu.edu/∼grant.
[T] M. Taylor, Introduction to Differential Equations, AMS 2011 (2nd ed. 2021).
[T2] M. Taylor, Linear Algebra, AMS, 2020.