Supplementary Material for Math 524 Text, Introduction to Differential Equations

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Introduction

These notes have arisen as a supplement to the text I have used in Math 524, *Introduction to Differential Equations.* There are several categories of items. Some beef up homework problems given in the text. Others present a different way to derive some specific result. Some merely correct some typos.

Chapter 1. Single differential equations

§1.3. Alternative endgame for the hanging cable problem.

We derived the differential equation

$$\frac{d\theta}{dx} = \beta \cos\theta, \tag{1.3.19}$$

which is separable,

$$\int \sec\theta \, d\theta = \int \beta \, dx = \beta x + \alpha. \tag{1.3.20}$$

As an alternative to (1.3.21), we can deduce instead from Exercise 14 of §1.1 that

$$\tan\theta(x) = \sinh(\beta x + \alpha). \tag{1.3.21$'$}$$

Again we have $\theta(0) = 0 \Rightarrow \alpha = 0$.

To get a formula for $y(x)$, use (1.3.14) to write

$$y(x) = y_0 + \int_0^x \tan\theta(t) \, dt, \quad y_0 = y(0). \tag{1.3.22}$$

This time, we can skip (1.3.23). Plugging in (1.3.21′) (with $\alpha = 0$) gives

$$\begin{aligned} y(x) &= y_0 + \int_0^x \sinh\beta t \, dt \\ &= y_0 - \frac{1}{\beta} + \frac{1}{\beta}\cosh\beta x. \end{aligned} \tag{1.3.24}$$

New exercise: finding β.

7. As derived in (1.3.25)–(1.3.26), the quantity β in (1.3.24) satisfies the equation

$$\beta = \frac{\tau}{a} > 0, \quad \frac{\sinh\tau}{\tau} = \frac{L}{a}. \tag{1.3.26}$$

Recall the positive quantities L and a are given and $L > a$. To show that for each $b = L/a \in (1,\infty)$ there is a unique $\tau > 0$ satisfying

$$h(\tau) = b, \quad \text{where } h(\tau) = \frac{\sinh\tau}{\tau}, \tag{A}$$

it is stated that

$$h(0) = 1, \quad h'(\tau) > 0 \text{ for } \tau > 0, \quad h(\tau) \nearrow +\infty \text{ as } \tau \nearrow +\infty. \tag{B}$$

Verify these statements. Show how they lead to a unique solution to (A).
Hint. Bring in the mean value theorem and the intermediate value theorem, from calculus.
Using a computer or calculator, find an approximate solution to (A) when $b = 2$. Investigate other values of b.

§1.16. New exercise: formulas for the Bessel functions $Y_n(t)$.

7. Write (1.16.24) as

$$J_\nu(t)=\sum_{k=0}^\infty \alpha_k(\nu)\Bigl(\frac{t}{2}\Bigr)^{2k+\nu},\quad \alpha_k(\nu)=\frac{(-1)^k}{k!\Gamma(k+\nu+1)}.$$

Show that (1.16.36) leads to

$$Y_0(t)=\frac{2}{\pi}\Bigl(\log\frac{t}{2}\Bigr)J_0(t)+\frac{2}{\pi}\sum_{k=0}^\infty \alpha_k'(0)\Bigl(\frac{t}{2}\Bigr)^{2k},$$

and, for $n\in\mathbb{N}$,

$$\begin{aligned}Y_n(t)=\frac{2}{\pi}\Bigl(\log\frac{t}{2}\Bigr)J_n(t)&+\frac{(-1)^n}{\pi}\sum_{k=0}^\infty \alpha_k'(-n)\Bigl(\frac{t}{2}\Bigr)^{2k-n}\\ &+\frac{1}{\pi}\sum_{k=0}^\infty \alpha_k'(n)\Bigl(\frac{t}{2}\Bigr)^{2k+n}.\end{aligned}$$

§1.17. Independence of exponential polynomials.

The following exercises complement Exercises 4–6. Let

$$\mathcal{E}_\lambda = \{e^{\lambda t} p(t) : p(t) \text{ polynomial}\}.$$

7. Show that

$$\lambda \neq 0 \Longrightarrow \partial_t = \frac{d}{dt} : \mathcal{E}_\lambda \to \mathcal{E}_\lambda \text{ is injective.} \tag{1}$$

(In fact, it is an isomorphism.)

8. Assume $K \in \mathbb{N}$ and $\{\lambda_1, \dots, \lambda_K\}$ are distinct. Let $p_j(t)$ be polynomials. Show that

$$e^{\lambda_1 t} p_1(t) + \cdots + e^{\lambda_K t} p_K(t) \equiv 0 \Longrightarrow \text{ each } p_j = 0. \tag{2}$$

Hint. Use induction on K. The case $K = 1$ is easy. Say $K \geq 2$ and the hypothesis in (2) holds. Multiply by $e^{-\lambda_K t}$ and relabel the exponents, to arrange that $\lambda_K = 0$. Then

$$e^{\lambda_1 t} p_1(t) + \cdots + e^{\lambda_k t} p_k(t) = -p_K(t), \quad k = K - 1.$$

We now have $\{0, \lambda_1, \dots, \lambda_k\}$ distinct. Apply ∂_t^N with $N > \deg p_K$. Use the inductive hypothesis in concert with (1) to show that each $p_j = 0$.

Chapter 2. Linear algebra

§2.7. Short proof of generalized eigenspace decomposition.

Let V be a complex vector space, $\dim V = n < \infty$, and take $T \in \mathcal{L}(V)$. The roots $\{\lambda_j\}$ of $\det(\lambda I - T)$ are the eigenvalues of T. We set

$$\mathcal{GE}(T, \lambda_j) = \{v \in V : (T - \lambda_j)^k v = 0 \text{ for some } k\}. \tag{1}$$

We aim to give a proof that

$$V = \bigoplus_j \mathcal{GE}(T, \lambda_j), \tag{2}$$

different from (and shorter than) that done in the text. We follow the elegant argument in Lecture 9 of [G], with a few tweaks.

Start with generalities. For $A \in \mathcal{L}(V)$, set

$$\mathcal{N}^{\#}(A) = \bigcup_k \mathcal{N}(A^k), \quad \mathcal{R}^{\#}(A) = \bigcap_k \mathcal{R}(A^k), \tag{3}$$

where $\mathcal{N}(A)$ is the null space of A and $\mathcal{R}(A)$ is its range. We have stabilization: for some m,

$$\mathcal{N}^{\#}(A) = \mathcal{N}(A^m) = \mathcal{N}(S), \quad \mathcal{R}^{\#}(A) = \mathcal{R}(A^m) = \mathcal{R}(S), \tag{4}$$

where $S = A^m$. Note that

$$A, S : \mathcal{R}(S) \longrightarrow \mathcal{R}(S) \text{ are onto, hence isomorphisms.} \tag{5}$$

Lemma 1. *We have*

$$V = \mathcal{N}(S) \oplus \mathcal{R}(S) = \mathcal{N}^{\#}(A) \oplus \mathcal{R}^{\#}(A). \tag{6}$$

Proof. Proposition 2.3.6 (the fundamental theorem of linear algebra) gives

$$\dim V = \dim \mathcal{N}(S) + \dim \mathcal{R}(S), \tag{7}$$

so it suffices to note that, when (5) holds,

$$\mathcal{N}(S) \cap \mathcal{R}(S) = \{0\}. \tag{8}$$

We apply this to

$$A = T - \lambda_j I, \tag{9}$$

where we pick an eigenvalue λ_j of T. So (6) says

$$V = \mathcal{GE}(T, \lambda_j) \oplus \mathcal{R}^{\#}(T - \lambda_j I), \tag{10}$$

and (5) implies that

$$T - \lambda_j I : \mathcal{R}^{\#}(T - \lambda_j I) \xrightarrow{\approx} \mathcal{R}^{\#}(T - \lambda_j I), \tag{11}$$

and consequently

$$T : \mathcal{R}^{\#}(T - \lambda_j I) \longrightarrow \mathcal{R}^{\#}(T - \lambda_j I). \tag{12}$$

We are ready for the main result (compare Propositions 2.7.5–2.7.6):

Theorem 2. *The direct decomposition (2) holds.*

Proof. Use induction on $\dim V$. If λ_j is an eigenvalue, then $\dim \mathcal{GE}(T,\lambda_j) \ge 1$, so (10)–(12) hold, with $\dim \mathcal{R}^{\#}(T-\lambda_j I) < \dim V$. Inductively,

$$\mathcal{R}^{\#}(T-\lambda_j I) = \bigoplus_{k\neq j} \mathcal{GE}(T,\lambda_k), \tag{13}$$

and we are done.

Remark. We have

$$\det(\lambda I - T) = \prod_j \det(\lambda I - T_j), \quad T_j = T\big|_{\mathcal{GE}(T,\lambda_j)} \in \mathcal{L}(\mathcal{GE}(T,\lambda_j)), \tag{14}$$

and each $T_j = \lambda_j I + N_j$, with N_j nilpotent, so $\mathcal{GE}(T,\lambda_j)$ has a basis in which the matrix of N_j is strictly upper triangular. (See §2.8.) Hence

$$\det(\lambda I - T_j) = (\lambda - \lambda_j)^{d_j}, \quad d_j = \dim \mathcal{GE}(T,\lambda_j). \tag{15}$$

Remark 2. Generally, if N is nilpotent on a vector space of dimension d, then $N^d = 0$. In particular, $(T-\lambda_j I)^{d_j}|_{\mathcal{GE}(T,\lambda_j)} = 0$. Hence if $K_T(\lambda) = \det(\lambda I - T)$ denotes the characteristic polynomial of T, we have from (14)–(15), together with (2), that

$$K_T(T) = \prod_j (T-\lambda_j I)^{d_j} = 0, \tag{16}$$

which is the Cayley-Hamilton theorem. Compare the proof of (2.8.17).

Reference

[G] C. Grant, Theory of Ordinary Differential Equations, Lecture Notes for Math 634, Brigham Young Univ., available at http://www.math.byu.edu/∼grant.

§2.11. Determinant criterion for positive definiteness.

Recall that $A \in M(n, \mathbb{C})$ is positive definite if and only if $A = A^*$ and $(Av, v) > 0$ for all nonzero v. We establish the following criterion.

Proposition 2.11.5. *Assume* $A = A^* \in M(n, \mathbb{C})$. *For* $1 \le k \le n$, *let* $A_k \in M(k, \mathbb{C})$ *denote the upper left* $k \times k$ *corner of* A *(so* $A_n = A$*). Then* A *is positive definite if and only if*

$$\det A_k > 0, \quad \forall\, k \in \{1, \dots, n\}. \tag{2.11.24}$$

Proof. Clearly (2.11.24) holds if A is positive definite. We need to prove that (2.11.24) implies A is positive definite. We proceed by induction on dimension, the case of dimension 1 being obvious. Assume we have the result in dimension $n - 1$. Then the hypothesis (2.11.24) implies A_{n-1} is positive definite. Now suppose A is not positive definite. Then A must have at least 1 negative eigenvalue, and hence, since $\det A > 0$, it must have at least 2. Let V denote the span of $\{\mathcal{E}(A, \lambda_j) : \lambda_j < 0\}$. We have $\dim V \ge 2$. If $E_{n-1} = \mathrm{Span}\{e_1, \dots, e_{n-1}\}$ (e_j denoting the standard basis of $\mathbb{C}^n$), then

$$E_{n-1} \cap V \ne 0.$$

We have

$$0 \ne v \in E_{n-1} \cap V \Rightarrow (Av, v) > 0 \text{ and } (Av, v) < 0,$$

the first result because A_{n-1} is positive definite. This is a contradiction, and the induction argument is complete.

§2.12. Addition to Exercise 9.

Show that if $T \in SO(3)$ and $\kappa : \mathbb{R}^3 \to \mathrm{Skew}(3)$ is given by $\kappa(y)x = y \times x$, then

$$\kappa(Ty) = T\kappa(y)T^{-1}.$$

Chapter 3. Linear systems of differential equations

§3.1. Further matrix exponential exercises.

15. Say $A \in M(n, \mathbb{C})$ belongs to $\mathrm{Skew}(\mathbb{C}^n)$ provided $A^* = -A$. Recall the definitions of $U(n)$ and $SU(n)$ from §2.12. Show that

$$A \in \mathrm{Skew}(\mathbb{C}^n) \Longrightarrow e^{tA} \in U(n), \quad \forall\, t \in \mathbb{R},$$

and

$$A \in \mathrm{Skew}(\mathbb{C}^n),\ \mathrm{Tr}\, A = 0 \Longrightarrow e^{tA} \in SU(n), \quad \forall\, t \in \mathbb{R}.$$

16. Establish converses of the implications in Exercise 15.

17. Apply d/dt to show that, for $A, B \in M(n, \mathbb{C})$,

$$AB = BA \Longrightarrow e^{-tB} A e^{tB} = A, \quad \forall\, t \in \mathbb{R},$$

hence obtaining an alternative derivation of (3.1.15).

§3.2. New exercise: exponentials of cross product operators.

7. Consider $\kappa : \mathbb{R}^3 \to \mathrm{Skew}(3)$, given by $\kappa(y)x = y \times x$, discussed in Exercise 9 of §2.12. Denoting the standard basis of $\mathbb{R}^3$ as $\{i, j, k\}$, show that

$$\kappa(k) = \begin{pmatrix} J & \\ & 0 \end{pmatrix}, \quad \text{hence} \quad e^{t\kappa(k)} = \begin{pmatrix} e^{tJ} & \\ & 1 \end{pmatrix},$$

for all $t \in \mathbb{R}$, with e^{tJ} given by (3.2.8). Take $T \in SO(3)$. Show that

$$e^{t\kappa(Ty)} = T e^{t\kappa(y)} T^{-1},$$

hence

$$y = Tk \Longrightarrow e^{t\kappa(y)} = T \begin{pmatrix} e^{tJ} & \\ & 1 \end{pmatrix} T^{-1}.$$

§3.11. Regular singular points: complementary results.

This section studies linear systems of the form

$$t\frac{dx}{dt} = A(t)x, \tag{3.11.1}$$

when x takes values in $\mathbb{C}^n$, $A(t)$ in $M(n,\mathbb{C})$, and $A(t)$ is given by a convergent power series, $A(t) = A_0 + A_1 t + A_2 t^2 + \cdots$, for $|t| < T_0$. We recall the first key result:

Lemma 3.11.1. *If $v \in \mathcal{N}(A_0)$ and if no positive integer is an eigenvalue of A_0, then (3.11.1) has a solution $x(t)$ given by a convergent power series, $x(t) = \sum_{k\ge 0} x_k t^k$, with $x_0 = v$.*

Here is a useful extension.

Proposition 3.11.1A. *Assume*

$$\lambda \in \operatorname{Spec}(A_0), \quad v \in \mathcal{E}(A_0,\lambda). \tag{11.A.1}$$

Assume

$$\lambda + k \notin \operatorname{Spec}(A_0), \quad \text{for} \ \ k \in \mathbb{N}.$$

Then the system (3.11.1) has a solution of the form

$$x(t) = t^\lambda y(t), \tag{11.A.2}$$

for small $t > 0$, where

$$y(t) = \sum_{k=0}^\infty y_k t^k, \quad y_0 = v. \tag{11.A.3}$$

Proof. If we set $x(t) = t^\lambda y(t),\ t > 0$, then (3.11.1) is converted to

$$t\frac{dy}{dt} = (A(t) - \lambda)y, \tag{11.A.4}$$

and Lemma 3.1.11 applies, with $A(t)$ replaced by $A(t) - \lambda$.

As noted in the text, Bessel's equation converts to a 2×2 system of the form (3.11.1), with

$$A(t) = A_0 + A_2 t^2, \quad A_0 = \begin{pmatrix} 0 & 1 \\ \nu^2 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}. \tag{11.A.5}$$

We have $\operatorname{Spec}(A_0) = \{\nu, -\nu\}$. In such a case, Proposition 3.11.1A yields 2 linearly independent solutions as long as $\nu > 0$ and 2ν is not an integer. As noted in the text, this result is weaker than results on Bessel's equation obtained in §1.16. This observation motivates the following result.

Lemma 3.11.1B. *In the setting of Lemma 3.11.1, assume that* $A(t)$ *is an even function of* t*. Assume* $v \in \mathcal{N}(A_0)$*, and that* A_0 *has no eigenvalues that are positive even integers. Then the system (3.11.1) has a solution about the origin,* $x(t) = \sum_{k\ge 0} x_{2k}t^{2k}$*, satisfying* $x_0 = v$*.*

The proof is a simple variant of the proof of Lemma 3.11.1.

This result in turn leads to the following variant of Proposition 3.11.1A.

Proposition 3.11.1C. *In the setting of Proposition 3.11.1A. assume* $A(t)$ *is even in* t*,* $\lambda \in \operatorname{Spec} A_0$*, and* $v \in \mathcal{E}(A_0, \lambda)$*. Also assume*

$$\lambda + 2k \notin \operatorname{Spec} A_0, \quad \textit{for} \ \ k \in \mathbb{N}.$$

Then the system (3.11.1) has a solution of the form (11.A.2), with $y(t)$ *as in (11.A.3).*

Proposition 3.11.1C applies to the Bessel system, where $A(t)$ is given by (11.A.5), to produce a 2D space of solutions on $(0, \infty)$, if $\nu \in \mathbb{C}\setminus\mathbb{Z}$, and a 1D space of solutions if $\nu = n \in \mathbb{Z}$ (which one verifies to be the same for $\nu = n$ and $\nu = -n$). By results of §1.16, this solution set is spanned by

$$\begin{pmatrix} J_\nu(t) \\ tJ_\nu'(t) \end{pmatrix} \ \text{and} \ \begin{pmatrix} J_{-\nu}(t) \\ tJ_{-\nu}'(t) \end{pmatrix}, \tag{11.A.6}$$

for $\nu \in \mathbb{C} \setminus \mathbb{Z}$. Results of §1.16 also yield, for $\nu = n \in \mathbb{Z}$, the spanning set

$$\begin{pmatrix} J_n(t) \\ tJ_n'(t) \end{pmatrix} \ \text{and} \ \begin{pmatrix} Y_n(t) \\ tY_n'(t) \end{pmatrix}. \tag{11.A.7}$$

A treatment of (3.11.1) that applies to Bessel's equation to yield a 2D solution set is given in Proposition 3.11.5, and the calculations (3.11.65)–(3.11.79). They show terms containing $\log t$ as a factor, when $\nu = n \in \mathbb{Z}$. See also the new Exercise 7 for §1.16. The treatment in §3.11 of the text shows that log terms arise by exponentiating certain matrices B that are not diagonalizable, or more precisely by evaluating $t^B = e^{B \log t}$.

Chapter 4. Nonlinear systems of differential equations

§4.7. From Lagrange's equations to Hamilton's equations.

Recall that if $L = L(x, v)$ is a Lagrangian, then Lagrange's equation is

$$\frac{d}{dt} L_v(x, x') = L_x(x, x'). \tag{7A.1}$$

Exercise 2 of this section says that if

$$E(x, v) = L_v(x, v)v - L(x, v), \tag{7A.2}$$

and if $x(t)$ solves (7A.1), then $(d/dt)E(x, x') = 0$. This is a conserved "energy."

It is natural to set

$$p = L_v(x, v), \tag{7A.3}$$

and consider the coordinate change

$$\psi(x, v) = (x, p). \tag{7A.4}$$

We have

$$D\psi(x, v) = \begin{pmatrix} I & 0 \\ L_{vx} & L_{vv} \end{pmatrix}. \tag{7A.5}$$

We see from the inverse function theorem that ψ is a local diffeomorphism provided

$$L_{vv}(x, v) \text{ is an invertible matrix.} \tag{7A.6}$$

We assume (7A.6) holds. We then express the energy in (x, p) coordinates as

$$\mathcal{E}(x, p) = E(x, v), \tag{7A.7}$$

that is,

$$E(x, v) = \mathcal{E}(x, L_v(x, v)). \tag{7A.8}$$

We aim to establish the following.

Proposition 7A.1. *The Lagrange equation (7A.1) is equivalent to the Hamiltonian system*

$$\frac{dx}{dt} = \frac{\partial \mathcal{E}}{\partial p}, \quad \frac{dp}{dt} = -\frac{\partial \mathcal{E}}{\partial x}. \tag{7A.9}$$

To start, we write the Lagrange system as

$$\frac{dx}{dt} = v, \quad \frac{dp}{dt} = L_x(x, v). \tag{7A.10}$$

It remains to compare this with (7A.9). Now differentiating (7A.8) gives

$$\begin{aligned} \frac{\partial E}{\partial x} &= \frac{\partial \mathcal{E}}{\partial x} + \frac{\partial \mathcal{E}}{\partial p} L_{vx}(x, v), \\ \frac{\partial E}{\partial v} &= \frac{\partial \mathcal{E}}{\partial p} L_{vv}(x, v). \end{aligned} \tag{7A.11}$$

Meanwhile, (7A.2) gives

$$\begin{aligned} \frac{\partial E}{\partial x} &= L_{vx}(x, v)v - L_x(x, v), \\ \frac{\partial E}{\partial v} &= L_{vv}(x, v)v. \end{aligned} \tag{7A.12}$$

Consequently the second identities in (7A.11)–(7A.12) give

$$\frac{\partial \mathcal{E}}{\partial p} = v, \tag{7A.13}$$

under the hypothesis (7A.6). Then comparing the first identities in (7A.11)–(7A.12) gives

$$\begin{aligned} \frac{\partial \mathcal{E}}{\partial x} &= \frac{\partial E}{\partial x} - vL_{vx}(x, v), \\ &= L_{vx}(x, v)v - L_x(x, v) - vL_{vx}(x, v) \\ &= -L_x(x, v). \end{aligned} \tag{7A.14}$$

Hence (7A.10) leads to the Hamiltonian system (7A.9).

Example. Consider the Lagrangian

$$L(x,v)=T-V=\frac{m}{2}v\cdot G(x)v-V(x), \tag{7A.15}$$

for $x\in\Omega\subset\mathbb{R}^n,\ v\in\mathbb{R}^n$, where $G(x)\in M(n,\mathbb{R})$ is symmetric and positive definite, introduced in Exercise 2 of §4.7. By (7A.2), the energy is

$$E(x,v)=\frac{m}{2}v\cdot G(x)v+V(x). \tag{7A.16}$$

The formula (7A.3) for p becomes

$$p=mG(x)v, \tag{7A.17}$$

hence

$$v=\frac{1}{m}H(x)p,\quad H(x)=G(x)^{-1}, \tag{7A.18}$$

and the energy $\mathcal{E}(x,p)$ in (x,p) coordinates becomes

$$\mathcal{E}(x,p)=\frac{1}{2m}p\cdot H(x)p+V(x). \tag{7A.19}$$

Hence the Hamiltonian system (7A.9) takes the form

$$\frac{dx}{dt}=\frac{1}{m}H(x)p,\quad \frac{dp_k}{dt}=\frac{1}{2m}p\cdot\frac{\partial H}{\partial x_k}p+\frac{\partial V}{\partial x_k}. \tag{7A.20}$$

Compare the approach in Exercise 7 of §4.7, whose setting was restricted to that of (7A.15), rather than the general case treated in (7A.1)–(7A.14). Further results on this special case, whose energy functions (7A.19) are called *momentum-quadratic Hamiltonians*, are pursued in §4.10.

Further results on constrained variational problems.

In (4.7.21)–(4.7.33) we considered variational problems for

$$I(u)=\int_a^b L(u(t),u'(t))\,dt \tag{7B.1}$$

subject to the constraint that

$$u:[a,b]\longrightarrow M\subset\mathbb{R}^n, \tag{7B.2}$$

an $(n-1)$-dimensional surface. (As before, $u(a)=p,\ u(b)=q$.) We obtained in (4.7.24) the modified Lagrange equation

$$\frac{d}{dt}L_v(u(t),u'(t))-L_x(u(t),u'(t))=a(t)n(u(t)), \tag{7B.3}$$

with real valued $a(t)$ as yet unevaluated. We worked out the ODE (4.7.33), for the special case (4.7.26), i.e., $L(x,v)=(1/2)\|v\|^2$. Here we consider the more general situation.

As before, we make use of (7B.2) to write $u'(t)\cdot n(u(t))=0$, hence

$$u''(t)\cdot n(u(t))=-u'(t)\cdot\frac{d}{dt}n(u(t)). \tag{7B.4}$$

Also, we rewrite (7B.3) as

$$L_{vv}(u,u')u''+L_{vx}(u,u')-L_x(u,u')=a(t)n(u(t)). \tag{7B.5}$$

Now, with

$$P_M^\perp w=(w\cdot n)n,\quad P_M w=(I-P_M^\perp)w, \tag{7B.6}$$

where $P_M=P_M(u(t))$, we have

$$\begin{aligned}&P_M[L_{vv}(u,u')u''+L_{vx}(u,u')u'-L_x(u,u')]=0,\\&P_M^\perp u''=-\Big[u'\cdot\frac{d}{dt}n(u)\Big]n(u),\end{aligned} \tag{7B.7}$$

We write this system schematically as

$$\begin{aligned}&P_M L_{vv}(u,u')u''=P_M A(u,u'),\\&P_M^\perp u''=P_M^\perp B(u,u').\end{aligned} \tag{7B.8}$$

Going a step further, we note that

$$\begin{aligned} P_M L_{vv}(u,u')u'' &= P_M L_{vv}(u,u')P_M u'' + P_M L_{vv}(u,u')P_M^{\perp} u'' \\ &= P_M L_{vv}(u,u')P_M u'' + P_M L_{vv}(u,u')P_M^{\perp} B(u,u'), \end{aligned} \tag{7B.9}$$

and rewrite (7B.8) as

$$\begin{aligned} &P_M L_{vv}(u,u')P_M u'' = P_M[A(u,u') - L_{vv}(u,u,)P_M^{\perp} B(u,u')] \\ &P_M^{\perp} u'' = P_M^{\perp} B(u,u'). \end{aligned} \tag{7B.10}$$

Then the equations combine to give

$$[P_M(u)L_{vv}(u,u')P_M(u) + P_M^{\perp}(u)]u'' = \Phi(u,u'). \tag{7B.11}$$

This is a well posed second order system of ODEs for u as long as

$$P_M(u)L_{vv}(u,u')P_M(u) + P_M^{\perp}(u) \ \text{ is invertible,} \tag{7B.12}$$

as an $n \times n$ matrix. In particular, this holds as long as

$$L_{vv}(u,u') \text{ is positive definite,} \tag{7B.13}$$

as an element of $M(n,\mathbb{R})$.

§4.9. Further double pendulum exercises.

1A. Show that the second equation in (4.9.35) can be rewritten as

$$\text{(A)} \qquad \theta_2'' + \frac{g}{\ell_2}\sin\theta_2 = \frac{\ell_1}{\ell_2}(\theta_1')^2\sin(\theta_1-\theta_2) + \frac{g}{\ell_2}(\sin\theta_1)\cos(\theta_1-\theta_2).$$

1B. Assume $\theta_1(t) = \varepsilon\sin\omega t$, mod $O(\varepsilon^2)$, with $\omega = \sqrt{g/\ell_1}$. Show that (A) becomes, mod $O(\varepsilon^2)$,

$$\text{(B)} \qquad \theta_2'' + \frac{g}{\ell_2}\sin\theta_2 = \varepsilon\frac{g}{\ell_2}(\cos\theta_2)\sin\omega t.$$

1C. If also $\theta_2 = \varepsilon\vartheta_2$, mod $O(\varepsilon^2)$, show that one gets

$$\text{(C)} \qquad \vartheta_2'' + \frac{g}{\ell_2}\vartheta_2 = C\sin\omega t, \quad \omega = \sqrt{\frac{g}{\ell_1}},$$

mod $O(\varepsilon^2)$. Note that (C) displays a *resonance* at $\ell_1 = \ell_2$.

1D. Peek ahead at the exercises following §4.15 to see more fun things you can do, especially with (B).

§4.10. Corrected proof of Lemma 4.10.2.

We provide a new proof of the following:

Lemma 4.10.2. *Let* $W \in M(n, \mathbb{R})$ *be symmetric, and assume*

$$W \text{ has } k \text{ positive and } n-k \text{ negative eigenvlaues.} \tag{4.10.15}$$

Then so has AWA, *when* $A \in M(n, \mathbb{R})$ *is positive definite.*

Proof. Since $\det AWA > 0$, each eigenvalue is nonzero. Assume AWA has κ positive eigenvalues and $n - \kappa$ negative eigenvalues. Let V denote $\operatorname{Span}\{\mathcal{E}(AWA, \lambda_j) : \lambda_j > 0\}$. Then

$$v \in V \Rightarrow 0 \leq AWAv \cdot v = WAv \cdot Av, \tag{4.10.16}$$

so

$$w \in AV \Longrightarrow Ww \cdot w \geq 0. \tag{4.10.17}$$

Let $N = \operatorname{Span}\{\mathcal{E}(W, \mu_j) : \mu_j < 0\}$, so $\dim N = n - k$. We have

$$\dim V > k \Rightarrow \dim AV > k \Rightarrow AV \cap N \neq 0, \tag{4.10.18}$$

but

$$0 \neq x \in AV \cap N \Rightarrow Wx \cdot x > 0 \text{ and } Wx \cdot x < 0. \tag{4.10.19}$$

Contradiction. Hence $\kappa \leq k$. A similar argument gives $n - \kappa \leq n - k$, hence $\kappa = k$.

4.13. Hopf bifurcation in a predator-prey model.

One family of predator-prey equations treated in this section has the form

$$
\begin{aligned}
x' &= -ax + b\zeta(y)x, \\
y' &= ry(1 - cy) - \zeta(y)x,
\end{aligned} \tag{1}
$$

with

$$
\zeta(y) = \frac{\kappa y}{1 + \gamma y}, \tag{2}
$$

where all parameters $a, b, c, r, \kappa, \gamma$ are positive. Here $x(t) = \#$ predators, $y(t) = \#$ prey, and $\zeta(y)$ is the rate at which each predator consumes prey. The term $ry(1-cy)$ in the second line of (1) arises as follows. In the absence of predators, we get $y' = ry(1 - cy)$. If $c = 0$, this models exponential population growth. For $c > 0$, the factor $(1 - cy)$ yields a logistic equation, enforcing $y = 1/c$ as the upper limit of the sustainable population y.

In Figure 4.13.6, we have

$$
a = 1,\ b = 2,\ r = 1,\ \kappa = 1,\ \gamma = 1, \tag{3}
$$

and $c = 1/4$. In this case, the critical point denoted (x_0, y_0) for the vector field X defined by (1) is a spiral source.

As indicated in Exercise 5 of this section, with the parameters (3), the critical point (x_0, y_0) is a source for $c \in (0, 1/3)$ and a sink for $c \in (1/3, 1)$, and at $c = 1/3$ the linearization of X at (x_0, y_0) is a center. As c decreases from $c > 1/3$ to $c < 1/3$, the qualitative behavior of the orbits undergoes a change, spitting out a periodic orbit in passage, as dictated by the Poincaré-Bendixson theorem. This transition is called a

Hopf bifurcation.

An excellent place to find out more about this phenomenon is the following:

J. Marsden and M. McCraken, The Hopf Bifurcation and its Applications, Springer NY, 1976.