Supplementary Material for the Text,
Introduction to Analysis in Several Variables
Pure and Applied Undergraduate Texts, #46

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Introduction

These notes have arisen as a supplement to the text I have used in Math 522, *Introduction to Analysis in Several Variables – Advanced Calculus.* There are several categories of items. Some beef up homework problems given in the text. Others present a different way to derive some specific result. Some merely correct some typos.

We highlight some of the supplements given in these notes. For Chapter 2, we provide a proof of the inverse function theorem based on a minimization argument, rather than the iterative construction used in the text. We also include a neat trick to show that if the inverse function theorem, applies to a C^1 map F, yielding a C^1 inverse G, then higher regularity $F \in C^k$ yields correspondingly higher regularity of G.

Supplementary material on the multidimensional integral in Chapter 3 includes some reworking of basic material, to bring to the fore a characterization of the upper and lower Riemann integrals $\overline{I}(f)$ and $\underline{I}(f)$ in terms of $\int g\,dV$, respectively for continuous $g \geq f$ and for continuous $g \leq f$. We use this to present a Fubini theorem that is much stronger and more general than Theorem 3.1.9, and furthermore has a slicker proof. We also present a generalization of Darboux's theorem to a setting of partitions of a cell by contented sets.

Two supplements to §3.2 discuss additional material on smooth maps between surfaces. We also explore the computation of further surface integrals, such as

$$\int\limits_{S^{n-1}} \omega^\alpha \, dS(\omega).$$

A supplement to §4.1 delves into the issue of defining a differential form directly on a surface $M \subset \mathbb{R}^n$, as an extension of the notion of a differential form defined on an open set in Euclidean space. We take the perspective that in an introduction to the theory of differential forms, it is desirable not to detour into the treatment of vector bundles, and instead define such forms as objects given in local coordinate systems, subject to natural compatibility conditions. (We do briefly indicate the bundle approach.)

A supplement to Chapter 5 extends the brief treatment of holomorphic functions on a domain in $\mathbb{C}$ given in §5.1. We derive the holomorphic inverse function theorem, as a corollary of the inverse function theorem of Chapter 2. Given that we have the real result, passing to the holomorphic case is a short step, much shorter than what one sees in standard complex analysis texts. We use this to establish the open mapping theorem for holomorphic maps.

Other supplements to Chapter 5 include further comments on the use of differential forms to prove the change of variable formula for integrals, and further results on the Euler characteristic of a compact 2D surface.

Chapter 6, on geometric properties of surfaces M in $\mathbb{R}^n$, brings in the family

$$P(x) = \text{orthogonal projection of } \mathbb{R}^n \text{ onto } T_x M$$

as an incisive tool. To supplement this, we provide some formulas for $P(x)$, extending well known formulas for 2D surfaces $M \subset \mathbb{R}^3$ that involve computing the unit normal $N(x)$.

Chapter 1. Background

§1.1. Sharper criterion for Riemann integrability

Here we aim to prove a Riemann integrability result that is sharper than Proposition 1.1.11. A key to this proof is the result that if $X \subset \mathbb{R}$ is compact, then each open cover of X has a finite subcover. This is a special case of Proposition 1.2.8 (in §1.2).

Proposition 1.1.11A. *Set $I = [a, b]$, and let $f : I \to \mathbb{R}$ be bounded, say $|f| \le M$. Suppose that the set S of points of discontinuity of f has the property*

$$m^*(S) = 0. \tag{1}$$

Then $f \in \mathcal{R}(I)$.

Proof. Fix $\varepsilon, \delta > 0$. Cover S with a countable family of open intervals J_ℓ such that $\sum_\ell \ell(J_\ell) \le \delta$. Set

$$K = I \setminus \bigcup_\ell J_\ell. \tag{2}$$

Then f is continuous at each $x \in K$, so for each $x \in K$, there is an open interval K_x, containing x, such that $|f(x) - f(y)| < \varepsilon$, for $y \in \overline{K}_x \cap I$. Hence $y, y' \in \overline{K}_x \cap I \Rightarrow |f(y) - f(y')| \le 2\varepsilon$.

Now $\{J_\ell, K_x\}$ is a cover of I by open intervals, so we can produce a finite subcover, $\{J_\ell, K_j\}$. Perhaps shrinking these intervals, and relabeling, we get a partition $\mathcal{P} = \{J_\ell, K_j\} = \mathcal{P}_0 \cup \mathcal{P}_1$ of I, with $\mathcal{P}_0 = \{J_\ell\}$, $\mathcal{P}_1 = \{K_j\}$. (These shrunken and relabeled intervals are now closed.) We have

$$\begin{aligned} J_\ell \in \mathcal{P}_0 &\Rightarrow \sup_{J_\ell} f - \inf_{J_\ell} f \le 2M, \quad \sum_\ell \ell(J_\ell) \le \delta, \\ K_j \in \mathcal{P}_1 &\Rightarrow \sup_{K_j} f - \inf_{K_j} f \le 2\varepsilon, \quad \sum_j \ell(K_j) \le b - a. \end{aligned} \tag{3}$$

Hence

$$\begin{aligned} \overline{I}_{\mathcal{P}}(f) - \underline{I}_{\mathcal{P}}(f) &\le \sum_{J_\ell \in \mathcal{P}_0} 2M\ell(J_\ell) + \sum_{K_j \in \mathcal{P}_1} 2\varepsilon\ell(K_j) \\ &\le 2M\delta + 2\varepsilon(b - a). \end{aligned} \tag{4}$$

This implies $f \in \mathcal{R}(I)$.

§1.1. Exercises on when $\int_a^b |f(x)|\,dx = 0$.

23. Let $f \in \mathcal{R}([a,b])$. Show that $\int_a^b |f(x)|\,dx = 0$ if and only if, for each $\varepsilon > 0$,

$$S_\varepsilon = \{x \in [a,b] : |f(x)| > \varepsilon\} \Longrightarrow \operatorname{cont}^+(S_\varepsilon) = 0.$$

In such a case,

$$S = \{x \in [a,b] : f(x) \neq 0\} \Longrightarrow m^*(S) = 0.$$

24. Let $f \in \mathcal{R}([0,1])$ be the function arising in Example 2, following Proposition 1.1.11. We have $\int_0^1 |f(x)|\,dx = 0$. Show that

$$S = \{x \in [0,1] : f(x) \neq 0\} = [0,1] \cap \mathbb{Q}, \quad \text{hence} \quad \operatorname{cont}^+ S = 1.$$

§1.1. The binomial formula

Here we establish the binomial formula,

$$(x+a)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} x^k, \quad \binom{n}{k} = \frac{n!}{k!(n-k)!}, \tag{1}$$

which plays a significant role in relating two multi-index notations in Chapter 2 (see (2.1.47)–(2.1.51)). The formula is used in some calculus courses to derive the formula

$$\frac{d}{dx} x^n = n x^{n-1}, \tag{2}$$

via

$$\frac{d}{dx} x^n = \lim_{a \to a} \frac{(x+a)^n - x^n}{a}. \tag{3}$$

However, we think it is easier to prove (2) using the product formula, obtaining

$$\frac{d}{dx} x^2 = \frac{d}{dx} x \cdot x = 2x, \quad \frac{d}{dx} x^3 = \frac{d}{dx} x \cdot x^2 = 3x^2, \tag{4}$$

and so on, getting (2) by induction on n. Hence, we take (2) as known, and use it as a tool to derive (1).

Clearly $(x+a)^n$ is a polynomial in x of degree n,

$$(x+a)^n = p(x) = b_n x^n + b_{n-1} x^{n-1} + \cdots + b_0. \tag{5}$$

Applying $(d/dx)^k$ and using (2), we see that

$$p^{(k)}(0) = k!\, b_k. \tag{6}$$

On the other hand, (2) also implies

$$\frac{d^k}{dx^k} (x+a)^n \Big|_{x=0} = n(n-1)\cdots(n-k+1)\, a^{n-k}. \tag{7}$$

Comparing (6)–(7) gives

$$b_k = \frac{n(n-1)\cdots(n-k+1)}{k!} a^{n-k} = \binom{n}{k} a^{n-k}, \tag{8}$$

and we have (1).

We mention Newton's extension of (1), to

$$(x+a)^r = \sum_{k=0}^{\infty} \binom{r}{k} a^{r-k} x^k, \quad |x| < |a|, \tag{9}$$

where

$$\binom{r}{k} = \frac{r(r-1)\cdots(r-k+1)}{k!}, \tag{10}$$

valid for $r \in \mathbb{R}$. See §4.3 of *Introduction to Analysis in One Variable.*

§1.4. Exercise relating determinant and trace.

5. Show that, for $A, B \in M(n, \mathbb{F})$,

$$\begin{aligned}\det(B + tA) &= \det B + t \sum_k \det(b_1, \dots, a_k, \dots, b_n) + O(t^2) \\ &= \det B + t \sum_{j,k} a_{jk} \operatorname{Cof}(B)_{jk} + O(t^2) \\ &= \det B + t \ \operatorname{Tr} \operatorname{Cof}(B)^t A + O(t^2).\end{aligned}$$

Comparing Exercise 4, re-derive Cramer's formula.

Chapter 2. Multivariable differential calculus

§2.1. Further power series exercises

1. Let $\Omega \subset \mathbb{R}^n$ be open, $f, g \in C^k(\Omega)$ real valued, $0 \in \Omega$. Write

$$f(x) = \sum_{|\beta| \le k} f_\beta x^\beta + o(x^k), \quad g(x) = \sum_{|\gamma| \le k} g_\gamma x^\gamma + o(x^k),$$

with

$$f_\beta = \frac{f^{(\beta)}(0)}{\beta!}, \quad g_\gamma = \frac{g^{(\gamma)}(0)}{\gamma!}.$$

Show that $h(x) = f(x)g(x)$ satisfies

$$h(x) = \sum_{|\beta|, |\gamma| \le k} f_\beta g_\gamma x^{\beta+\gamma} + o(x^k),$$

and deduce that, for $|\alpha| \le k$,

$$\frac{h^{(\alpha)}(0)}{\alpha!} = \sum_{\beta+\gamma=\alpha} f_\beta g_\gamma = \sum_{\beta+\gamma=\alpha} \frac{1}{\beta!\gamma!} f^{(\beta)}(0) g^{(\gamma)}(0).$$

Pass from this to the identity

$$\partial^\alpha (fg)(x) = \sum_{\beta+\gamma=\alpha} \frac{\alpha!}{\beta!\gamma!} f^{(\beta)}(x) g^{(\gamma)}(x), \tag{1}$$

for $x \in \Omega$. This identity is called the Leibniz identity.

2. Define $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x) = e^{x_1} \cos x_2.$$

Compute $f^{(\alpha)}(x)$ for $|\alpha| \le 3$. Then write down

$$P(x) = \sum_{|\alpha| \le 3} \frac{1}{\alpha!} f^{(\alpha)}(0) x^\alpha.$$

3. Attack the computation of $P(x)$ in Exercise 2 using Exercise 1, starting with

$$e^{x_1} = 1 + x_1 + \frac{x_1^2}{2} + \frac{x_1^3}{3!} + \cdots,$$

and a similar expansion of $\cos x_2$.

4. Write down the power series about $(0,0)$ of

$$F(x,y) = \int_0^1 \frac{e^{xt}}{1+yt}\, dt.$$

Hint. Start by multiplying the power series of e^{xt} and $(1+yt)^{-1}$.

5. Show that, for $x = (x_1, \dots, x_n)$, with $|x_j| < 1$ for all j,

$$\sum_{\alpha \geq 0} x^\alpha = \frac{1}{1-x_1} \cdots \frac{1}{1-x_n}.$$

Hint. Write the left side as

$$\sum_{\alpha_1 \geq 0} x_1^{\alpha_1} \cdots \sum_{\alpha_n \geq 0} x_n^{\alpha_n}.$$

6. In this exercise, we take

$$\eta = (t, t, \dots, t) \in \mathbb{R}^n, \quad |t| < 1,$$

and consider

$$F(\eta) = \sum_{\alpha \geq 0} \eta^\alpha.$$

(a) Show that, for $|t| < 1$,

$$F(\eta) = \sum_{\alpha_1 \geq 0} t^{\alpha_1} \cdots \sum_{\alpha_n \geq 0} t^{\alpha_n} = (1-t)^{-n}.$$

(b) Show that

$$F(\eta) = \sum_{\alpha \geq 0} t^{|\alpha|} = \sum_{k=0}^{\infty} d_k(n) t^k,$$

where

$$\begin{aligned} d_k(n) &= \#\{\alpha = (\alpha_1, \dots, \alpha_n) : |\alpha| = k\} \\ &= \dim \mathcal{P}_k(\mathbb{R}^n), \end{aligned}$$

with

$$\mathcal{P}_k(\mathbb{R}^n) = \text{ space of polynomials in } x \in \mathbb{R}^n, \text{ homogeneous of degree } k.$$

(c) Comparing results of (a) and (b), show that

$$\begin{aligned} d_k(n) &= \text{ coefficient of } t^k \text{ in } f_n(t) = (1-t)^{-n} \\ &= \frac{1}{k!} f_n^{(k)}(0) \\ &= \frac{n(n+1)\cdots(n+k-1)}{k!} \\ &= \binom{n+k-1}{k}. \end{aligned}$$

(d) If $\mathcal{P}^k(\mathbb{R}^n)$ = space of polynomials in $x \in \mathbb{R}^n$ of degree $\le k$, show that

$$\begin{aligned} \dim \mathcal{P}^k(\mathbb{R}^n) &= \dim \mathcal{P}_k(\mathbb{R}^{n+1}) \\ &= \binom{n+k}{k} = \binom{n+k}{n} \\ &= \frac{(k+n)(k+n-1)\cdots(k+1)}{n!}. \end{aligned}$$

(e) Another formulation of part (a) is that (with $\alpha = (\alpha_1, \dots, \alpha_n)$)

$$\sum_{\alpha \ge 0} t^{|\alpha|} = (1-t)^{-n}, \quad \text{for } \ |t| < 1.$$

Show how this leads to convergence in (2.1.86).

(f) Show that, for each $j \in \{1, \dots, n\}$,

$$\sum_{\alpha \ge 0} \alpha_j t^{|\alpha|} = t(1-t)^{-n-1}, \quad \text{for } \ |t| < 1.$$

Show how this leads to convergence in (2.1.91).

7. The multinomial theorem says

$$(x_1 + \cdots + x_n)^k = \sum_{|\alpha|=k} \binom{k}{\alpha} x^\alpha, \quad \binom{k}{\alpha} = \frac{k!}{\alpha!}, \tag{2}$$

where $\alpha! = \alpha_1! \cdots \alpha_n!$. Verify the following slick route to a proof. First,

$$e^{x_1 + \cdots + x_n} = \sum_{k=0}^{\infty} \frac{1}{k!}(x_1 + \cdots + x_n)^k, \tag{3}$$

second

$$\begin{aligned} e^{x_1} \cdots e^{x_n} &= \sum_{\alpha_1 \ge 0} \frac{x_1^{\alpha_1}}{\alpha_1!} \cdots \sum_{\alpha_n \ge 0} \frac{x_n^{\alpha_n}}{\alpha_n!} \\ &= \sum_{\alpha \ge 0} \frac{x^\alpha}{\alpha!}. \end{aligned} \tag{4}$$

Comparing (3) and (4) yields (2). Compare the derivation in (2.1.50)–(2.1.51).

§2.2. The inverse function theorem (alternative proof)

The inverse function theorem gives a condition under which a function can be locally inverted. This theorem and its corollary the implicit function theorem are fundamental results in multivariable calculus. First we state the inverse function theorem. Here, we assume $k \geq 1$.

Theorem 2.2.1. *Let F be a C^k map from an open neighborhood Ω of $p_0 \in \mathbb{R}^n$ to $\mathbb{R}^n$, with $q_0 = F(p_0)$. Suppose the derivative $DF(p_0)$ is invertible. Then there is a neighborhood $\mathcal{O}$ of p_0 and a neighborhood U of q_0 such that $F : \mathcal{O} \to U$ is one-to-one and onto, and $F^{-1} : U \to \mathcal{O}$ is a C^k map. (One says $F : \mathcal{O} \to U$ is a diffeomorphism.)*

We divide the task of proving this into two parts, first getting a one-to-one result, then getting a result about the image containing a certain neighborhood of q_0. Both of these results contain more quantitative information than is stated in Theorem 2.2.1, and are interesting in their own right.

Here is an injectivity result.

Proposition 2.2.2. *Assume $\Omega \subset \mathbb{R}^n$ is open and convex, and let $f : \Omega \to \mathbb{R}^n$ be C^1. Assume the symmetric part of $Df(u)$ is positive definite for each $u \in \Omega$. Then f is one-to-one on Ω.*

Proof. Take distinct points $u_1, u_2 \in \Omega$, and set $u_2 - u_1 = w$. Consider $\varphi : [0,1] \to \mathbb{R}$, given by

$$\varphi(t) = w \cdot f(u_1 + tw).$$

Then $\varphi'(t) = w \cdot Df(u_1 + tw)w > 0$ for $t \in [0,1]$, so $\varphi(0) \neq \varphi(1)$. But $\varphi(0) = w \cdot f(u_2)$ and $\varphi(1) = w \cdot f(u_2)$, so $f(u_1) \neq f(u_2)$.

Corollary 2.2.2A. *Take Ω and f as in the first sentence of Proposition 2.2.2, and assume there exists $\alpha < 1$ such that for all $x \in \Omega$,*

$$\|Df(x) - I\| \leq \alpha. \tag{2.1}$$

The f is one-to-one on Ω.

Proof. We can write

$$f(x) = x + R(x), \quad DR(x) = Df(x) - I, \quad \|DR(x)\| \leq \alpha, \tag{2.2}$$

so

$$w \cdot Df(x)w = \|w\|^2 - w \cdot DR(x)w \geq (1-\alpha)\|w\|^2, \tag{2.3}$$

and Proposition 2.2.2 applies. We obtain

$$(u_2 - u_1) \cdot [f(u_2) - f(u_1)] \geq (1-\alpha)\|u_2 - u_1\|^2. \tag{2.3A}$$

Second proof. For $x, y \in \Omega$,

$$x - y = \{f(x) - f(y)\} - \{R(x) - R(y)\}, \tag{2.4}$$

and the hypotheses imply

$$\|R(x) - R(y)\| \le \alpha \|x - y\|, \tag{2.5}$$

so

$$(1 - \alpha)\|x - y\| \le ||f(x) - f(y)||, \quad \forall\, x, y \in \Omega. \tag{2.6}$$

REMARK. The estimate (2.3A) implies (2.6), and in fact is more precise. One advantage of the second proof is that it extends to the Banach space setting, while the argument leading to (2.3A) requires a Hilbert space setting.

We now present a surjectivity result.

Proposition 2.2.3. *Let $\Omega \subset \mathbb{R}^n$ be open and bounded and contain p_0. Assume $F : \overline{\Omega} \to \mathbb{R}^n$ is continuous and $F(p_0) = q_0$. Assume F is C^1 on Ω and $DF(x)$ is invertible for all $x \in \Omega$. Finally, assume there exists $R > 0$ such that*

$$x \in \partial\Omega \Longrightarrow \|F(x) - q_0\| \ge R. \tag{2.7}$$

Then

$$F(\Omega) \supset B_{R/2}(q_0). \tag{2.8}$$

Proof. Given $y_0 \in B_{R/2}(q_0)$, we deduce from continuity of F and compactness of $\overline{\Omega}$ that there exists $x_0 \in \overline{\Omega}$ such that

$$\|F(x_0) - y_0\| = \inf_{x \in \overline{\Omega}} \|F(x) - y_0\|. \tag{2.9}$$

It follows from (2.7) and the fact that $\|F(p_0) - y_0\| < R/2$ that $x_0 \notin \partial\Omega$, hence $x_0 \in \Omega$. We claim that $F(x_0) = y_0$. Indeed, if $F(x_0) \ne y_0$, we can consider

$$F(x_0 + tz) = F(x_0) + tDF(x_0)z + o(\|tz\|), \tag{2.10}$$

with $z \in \mathbb{R}^n$ satisfying

$$DF(x_0)z = y_0 - F(x_0), \tag{2.11}$$

and deduce that, for small $t > 0$, $F(x_0 + tz)$ is closer to y_0 than $F(x_0)$ is. Contradiction, so $F(x_0) = y_0$.

A further short argument improves the conclusion of the last result.

Corollary 2.2.3A. *Under the hypotheses of Proposition 2.2.3, we actually have*

$$F(\Omega) \supset B_R(q_0). \tag{2.12}$$

Proof. It is convenient to translate coordinates, so $q_0 = 0$. It suffices to show that $F(\Omega) \supset B_S(0)$ for each $S \in (R/2, R)$. To get this, set $G(x) = \varphi \circ F(x)$, where $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ is a diffeomorphism satisfying

$$\varphi(y) = y \text{ for } |y| \le S, \quad \varphi(y) = 2y \text{ for } |y| \ge R. \tag{2.13}$$

Then we apply Proposition 2.2.3, with F replaced by G and R replaced by $2R$, to get $G(\Omega) \supset B_R(q_0)$, hence $F(\Omega) \supset B_S(q_0)$.

We can now put these results together to obtain a bijectivity result.

Proposition 2.2.4. *Let $p_0 \in \mathbb{R}^n$, $\rho > 0$, and $\Omega = B_\rho(p_0)$. Assume $F : \overline{\Omega} \to \mathbb{R}^n$ is continuous, and C^1 on Ω, and assume*

$$\|DF(x) - I\| \le \alpha, \quad \forall\, x \in \Omega, \tag{2.14}$$

with $\alpha < 1$. Let $F(p_0) = q_0$. Then F maps Ω one-to-one onto its image, and

$$F(\Omega) \supset B_R(q_0), \quad R = \rho(1 - \alpha). \tag{2.15}$$

Hence, taking

$$\mathcal{O} = F^{-1}(B_R(q_0)) \subset \Omega, \tag{2.16}$$

$\mathcal{O}$ is a open neighborhood of p_0 in $\mathbb{R}^n$, and

$$F : \mathcal{O} \longrightarrow B_R(q_0), \quad \textit{one-to-one and onto.} \tag{2.17}$$

Proof. That F maps Ω one-to-one onto its image follows from Corollary 2.2.2A. Also, (2.14) implies $DF(x)$ is invertible for each $x \in \Omega$, so Corollary 2.2.3A applies, and we have (2.15). Since $F(p_0) = q_0$ and F is continuous, $\mathcal{O}$ in (2.16) is an open set containing p_0, and (2.17) follows.

Remark. In the setting of Proposition 2.2.4, we have from (2.6) that

$$(1 - \alpha)\|x - x'\| \le \|F(x) - F(x')\|, \quad \forall\, x, x' \in \Omega, \tag{2.18}$$

hence for all $x, x' \in \mathcal{O}$, so the inverse map $G : B_R(q_0) \to \mathcal{O}$ satisfies

$$(1 - \alpha)\|G(y) - G(y')\| \le \|y - y'\|, \quad \forall\, y, y' \in B_R(q_0), \tag{2.19}$$

so G is continuous, in fact Lipschitz continuous, on $B_R(q_0)$.

This observation leads to the following differentiability result.

Proposition 2.2.5. *In the setting of Proposition 2.2.4, the inverse map* $G : B_R(q_0) \to \mathcal{O}$ *is differentiable at each* $y \in B_R(q_0)$*, and*

$$DG(y) = DF(x)^{-1}, \quad \text{for} \;\; y = F(x). \tag{2.20}$$

Proof. Given $x \in \mathcal{O},\ \xi \in \mathbb{R}^n$ small, we have

$$F(x+\xi) = F(x) + DF(x)\xi + R(x,\xi), \quad R(x,\xi) = o(|\xi|). \tag{2.21}$$

hence

$$\begin{aligned} G(F(x) + DF(x)\xi) &= G(F(x+\xi) - R(x,\xi)) \\ &= G \circ F(x+\xi) + O(R(x,\xi)) \\ &= x + \xi + o(|\xi|), \end{aligned} \tag{2.22}$$

the second identity by (2.19) and the rest by (2.21). Equivalently, given that $DF(x)$ is invertible, for $y = F(x) \in B_R(q_0)$, $\eta \in \mathbb{R}^n$ small,

$$G(y+\eta) = x + DF(x)^{-1}\eta + o(|\eta|), \tag{2.23}$$

yielding (2.20).

We are now ready for the

Proof of Theorem 2.2.1. Set

$$f(x) = AF(x), \quad A = DF(p_0)^{-1}, \tag{2.24}$$

so $Df(p_0) = I$. Then take $\alpha < 1$ and $\rho > 0$ such that $B_\rho(p_0) \subset \Omega$ and

$$\|Df(x) - I\| \le \alpha, \quad \forall\, x \in B_\rho(p_0). \tag{2.25}$$

Then Propositions 2.2.4–2.2.5 apply to f, so there exists an open set $\mathcal{O}$, containing p_0 and $R > 0$ such that

$$f : \mathcal{O} \longrightarrow B_R(Aq_0) \;\; \text{is one-to-one and onto,} \tag{2.26}$$

and the inverse map $g : B_R(Aq_0) \to \mathcal{O}$ is differentiable, with

$$Dg(y) = Df(x)^{-1}, \quad \text{for} \;\; x \in \mathcal{O},\ y = f(x). \tag{2.27}$$

It follows that, with $U = A^{-1}B_R(Aq_0)$, a neighborhood of q_0,

$$F : \mathcal{O} \longrightarrow U \;\; \text{is one-to-one and onto,} \tag{2.28}$$

with inverse $G : U \to \mathcal{O}$ that is differentiable, satisfying

$$DG(y) = DF(x)^{-1}, \quad \text{for} \;\; x \in \mathcal{O},\ y = F(x). \tag{2.29}$$

The formula

$$DG(y) = DF(G(y))^{-1} \tag{2.30}$$

presents DG as the composition of continuous maps, so $G \in C^1$.

Regarding higher differentiability of G in case $F \in C^k,\ k > 1$, we next take up an approach to this, different from that indicated in the text.

§2.2. Higher regularity of inverse functions

So far we have a proof of Theorem 2.2.1 for $F \in C^k$ when $k = 1$. If F has higher smoothness, one neat way to establish higher regularity for the inverse map G is to couple F and DF.

Thus we consider

$$\Phi : \Omega \times \mathbb{R}^n \longrightarrow \mathbb{R}^n \times \mathbb{R}^n, \tag{2.31}$$

given by

$$\Phi(x, v) = \begin{pmatrix} F(x) \\ DF(x)v \end{pmatrix}. \tag{2.32}$$

We retain the hypothesis that $DF(p_0)$ is invertible. Note that $F \in C^2 \Rightarrow \Phi \in C^1$ and

$$D\Phi(x, v) = \begin{pmatrix} DF(x) & 0 \\ D_x(DF(x)v) & DF(x) \end{pmatrix} \tag{2.33}$$

is invertible at (p_0, v). Hence we have a local C^1 inverse

$$\Psi(y, w) = \begin{pmatrix} G(y) \\ \psi(y, w) \end{pmatrix}, \tag{2.35}$$

where G is the local inverse of F. A calculation gives

$$\Phi(\Psi(y, w)) = \Phi\begin{pmatrix} G(y) \\ \psi(y, w) \end{pmatrix} = \begin{pmatrix} y \\ DF(G(y))\psi(y, w) \end{pmatrix}, \tag{2.35}$$

and since also $\Phi(\Psi(y, w)) = (y, w)^t$, we have $DF(G(y))\psi(y, w) = w$. Since $DF(G(y))$ $DG(y) = I$, we have $\psi(y, w) = DG(y)w$, hence the local inverse of Φ is

$$\Psi(y, w) = \begin{pmatrix} G(y) \\ DG(y)w \end{pmatrix}. \tag{2.36}$$

Now

$$F \in C^2 \Rightarrow \Phi \in C^1 \Rightarrow \Psi \in C^1 \Rightarrow G, DG \in C^1 \Rightarrow G \in C^2. \tag{2.37}$$

We can proceed by induction to establish the following.

Proposition 2.2.6. *In the setting of Theorem 2.2.1, the hypothesis $F \in C^k$ yields $G \in C^k$, for all $k \in \mathbb{N}$.*

Proof. To complete the induction, suppose we have the result for $k \in \{1, \dots, \ell\}$, and suppose $F \in C^{\ell+1}$. Again we have (2.31)–(2.36), and we can replace the chain of inferences in (2.37) by

$$F \in C^{\ell+1} \Rightarrow \Phi \in C^{\ell} \Rightarrow \Psi \in C^{\ell} \Rightarrow G, DG \in C^{\ell} \Rightarrow G \in C^{\ell+1}. \tag{2.38}$$

This does it.

Chapter 3. Multivariable integral calculus and calculus on surfaces

§3.1. Comments on Proposition 3.1.11

We concentrate on a part of Proposition 3.1.11.

Proposition 3.1.11A. *Given a cell $R \subset \mathbb{R}^n$ and $f : R \to \mathbb{R}$ bounded,*

$$\overline{I}(f) = \overline{I}_3(f) = \inf\Big\{ \int g\, dV : g \in C(R),\ g \geq f \Big\}, \tag{42A}$$

and

$$\underline{I}(f) = \underline{I}_3(f) = \sup\Big\{ \int g\, dV : g \in C(R),\ g \leq f \Big\}. \tag{43A}$$

Proof. We are skipping the parts of Proposition 3.1.11 that read

$$\overline{I}(f) = \overline{I}_1(f) = \overline{I}_2(f), \quad \underline{I}(f) = \underline{I}_1(f) = \underline{I}_2(f). \tag{1}$$

To begin the proof of (42A), first note that

$$\overline{I}_3(f) \geq \overline{I}(f). \tag{2}$$

This follows directly from the observation that, given $f, g : R \to \mathbb{R}$, bounded,

$$g \geq f \Longrightarrow \overline{I}(g) \geq \overline{I}(f), \tag{3}$$

so

$$g \in C(R),\ g \geq f \Longrightarrow \int_R g\, dV \geq \overline{I}(f). \tag{4}$$

We next tackle the converse to (2),

$$\overline{I}_3(f) \leq \overline{I}(f). \tag{5}$$

Adding a constant, we can assume $f \geq 0$ on R. Now pick $\varepsilon > 0$, and let $\mathcal{P} = \{R_\alpha\}$ be a partition of R such that

$$\overline{I}(f) \geq \overline{I}_{\mathcal{P}}(f) - \varepsilon = \sum_\alpha f_\alpha V(R_\alpha) - \varepsilon, \tag{6}$$

where

$$f_\alpha = \sup_{R_\alpha} f. \tag{7}$$

The key is to take

$$g_\alpha \in C(R), \quad g_\alpha \ge \chi_{R_\alpha}, \quad \int\limits_R g_\alpha \, dV \le (1+\varepsilon) V(R_\alpha). \tag{8}$$

Then we set

$$g = \sum_\alpha f_\alpha g_\alpha, \tag{9}$$

satisfying

$$g \in C(R), \quad g \ge f, \tag{10}$$

and

$$\begin{aligned} \int\limits_R g \, dV &\le \sum_\alpha f_\alpha (1+\varepsilon) V(R_\alpha) \\ &\le (1+\varepsilon)\Big[\overline{I}(f) + \varepsilon\Big]. \end{aligned} \tag{11}$$

This gives (5), and finishes the proof of (42A). The proof of (43A) is similar, so we have Proposition 3.1.11A.

Corollary 3.1.11B. *Given $f : R \to \mathbb{R}$ bounded, $f \in \mathcal{R}(R)$ if and only if there exist $\varphi, \psi \in C(R)$ such that*

$$\varphi \le f \le \psi, \quad \textit{and} \quad \int\limits_R (\psi - \varphi) \, dV < \varepsilon. \tag{12}$$

Remark. This result has a generalization, given in Lemma 3.1.16.

§3.1. Application to products

Using Proposition 3.1.11A, we can give another analysis of products of Riemann integrable functions.

Proposition 3.1.17. *If* $R \subset \mathbb{R}^n$ *is a cell, then*

$$f_1, f_2 \in \mathcal{R}(R) \Longrightarrow f_1 f_2 \in \mathcal{R}(R). \tag{1}$$

Proof. Looking at

$$(f_1 + a_1)(f_2 + a_2) = f_1 f_2 + a_1 f_2 + a_2 f_1 + a_1 a_2, \tag{2}$$

we see that there is no loss of generality in assuming $f_1, f_2 \geq 0$. Say also $f_j \leq M$. Now take $\varepsilon > 0$ and use Proposition 3.1.11A to pick φ_j, ψ_j such that

$$0 \leq \varphi_j \leq f_j \leq \psi_j \leq M, \quad \varphi_j, \psi_j \in C(R), \quad \int\limits_R (\psi_j - \varphi_j)\, dV < \varepsilon. \tag{3}$$

Then

$$\varphi_1 \varphi_2 \leq f_1 f_2 \leq \psi_1 \psi_2, \tag{4}$$

and

$$\begin{aligned} \psi_1 \psi_2 - \varphi_1 \varphi_2 &= \psi_1(\psi_1 - \varphi_2) + (\psi_1 - \varphi_1)\varphi_2 \\ &\leq M(\psi_2 - \varphi_2) + M(\psi_1 - \varphi_1), \end{aligned} \tag{5}$$

so

$$0 \leq \int (\psi_1 \psi_2 - \varphi_1 \varphi_2)\, dV \leq 2\varepsilon M, \tag{6}$$

and Proposition 3.1.11A implies $f_1 f_2 \in \mathcal{R}(R)$.

§3.1. Comments on Proposition 3.1.14

In the argument involving (3.1.48)–(3.1.51), it is convenient to impose on the partition $\mathcal{P} = \{R_\alpha\}$ the condition that the cells R_α be cubes, or at least that there be some a priori bound on the ratio of the sidelengths. This makes it straightforward to justify (3.1.19), as a consequence of the preceding formula for $G(\xi_\alpha + y)$. If the shortest sidelength of R_α were allowed to be tiny compared to the longest sidelength, this implication could fail.

Thanks to Mark Williams for pointing this out.

§3.1A. Further Fubini theorems

Here we present some Fubini theorems that are much stronger and more general than Theorem 3.1.9, and furthermore have a slicker proof. As a preliminary, we establish the following special case.

Proposition 1A.1. *Let $A \subset \mathbb{R}^k$, $B \subset \mathbb{R}^\ell$ be cells, $k + \ell = n$. Assume $f \in C(A \times B)$, and set*

$$g(x) = \int\limits_B f(x, y)\, dy. \tag{1A.1}$$

Then $g \in C(A)$ and

$$\int\limits_A g(x)\, dx = \int\limits_{A\times B} f\, dV. \tag{1A.2}$$

Proof. Since $A \times B$ is compact, f is uniformly continuous. Say $|f(z) - f(z')| \le \omega(|z - z'|)$. Then $|f(x, y) - f(x', y)| \le \omega(|x - x'|)$, so

$$|g(x) - g(x')| \le V_\ell(B)\omega(|x - x'|), \tag{1A.3}$$

hence $g \in C(A)$.

To proceed, pick $\varepsilon > 0$ and take a partition $\{R_{\alpha\beta}\} = \{R_\alpha \times R_\beta\}$ of $A \times B$, sufficiently fine that, for each α, β,

$$\operatorname{osc}_{R_{\alpha\beta}} f < \varepsilon, \quad \operatorname{osc}_{R_\alpha} g < \varepsilon, \tag{1A.4}$$

where $\operatorname{osc}_S f = \sup_S f - \inf_S f$. Pick $z_{\alpha\beta} = (x_\alpha, y_\beta) \in R_{\alpha\beta}$. Since

$$g(x) = \sum_\beta \int\limits_{R_\beta} f(x, y)\, dy, \tag{1A.5}$$

we have

$$g(x) = \sum_\beta f(z_{\alpha\beta}) V_\ell(R_\beta) + R_1(\varepsilon, x), \tag{1A.6}$$

for all $x \in R_\alpha$, with $|R_1(\varepsilon, x)| \le V_\ell(B)\varepsilon$. Furthermore,

$$\int\limits_A g(x)\, dx = \sum_\alpha g(x_\alpha) V_k(R_\alpha) + R_2(\varepsilon), \tag{1A.7}$$

with $|R_2(\varepsilon)| \le V_k(A)\varepsilon$. Then (1A.6)–(1A.7) yield

$$\int_A g(x)\,dx = \sum_{\alpha,\beta} f(z_{\alpha\beta}) V_k(R_\alpha) V_\ell(R_\beta) + R_3(\varepsilon), \tag{1A.8}$$

with $|R_3(\varepsilon)| \le [V_k(A) + V_n(A \times B)]\varepsilon$. On the other hand,

$$\int_{A\times B} f\,dV = \sum_{\alpha,\beta} f(z_{\alpha\beta}) V(R_{\alpha\beta}) + R_4(\varepsilon), \tag{1A.9}$$

with $|R_4(\varepsilon)| \le V(A\times B)\varepsilon$. Comparing (1A.8) with (1A.9) gives the asserted result (1A.2).

We proceed to the main result (advertised in Exercise 22).

Theorem 1A.2. *Let A, B be cells, as above, and take*

$$f \in \mathcal{R}(A \times B). \tag{1A.10}$$

Let $f_x(y) = f(x,y)$, so $f_x : B \to \mathbb{R}$, for $x \in A$. Set

$$g^+(x) = \overline{I}_B f_x, \quad g^-(x) = \underline{I}_B f_x. \tag{1A.11}$$

Then $g^\pm \in \mathcal{R}(A)$ and

$$\int_A g^\pm(x)\,dx = \int_{A\times B} f\,dV. \tag{1A.12}$$

Proof. We will derive this from Proposition 1A.1, via Proposition 3.1.11. Denote the right side of (1A.12) by $I(f)$. Take $\varepsilon > 0$. Then, using Proposition 3.1.11, pick $\varphi, \psi \in C(A \times B)$ such that

$$\varphi \le f \le \psi, \quad \text{and} \quad \int_{A\times B} (\psi - \varphi)\,dV < \varepsilon. \tag{1A.13}$$

We have $\varphi_x \le f_x \le \psi_x$ for each $x \in A$, so

$$\int_B \varphi_x(y)\,dy \le g^-(x) \le g^+(x) \le \int_B \psi_x(y)\,dy, \tag{1A.14}$$

for all $x \in A$. Proposition 1A.1 gives

$$\int_A \Bigl(\int_B \varphi(x,y)\,dy\Bigr) dx = \int_{A\times B} \varphi\,dV, \tag{1A.15}$$

and similarly for ψ. Hence

$$\int\limits_{A\times B} \varphi\, dV \le \underline{I}_A g^- \le \overline{I}_A g^+ \le \int\limits_{A\times B} \psi\, dV. \tag{1A.16}$$

Taking $\varepsilon \searrow 0$ gives

$$I(f) \le \underline{I}_A g^- \le \overline{I}_A g^+ \le I(f). \tag{1A.17}$$

Hence equality holds at each step of (1A.17). It follows from this that $g^\pm \in \mathcal{R}(A)$ and (1A.12) holds.

Remark. We also have

$$I_A(g^+ - g^-) = 0 \Longrightarrow g^+(x) = g^-(x), \quad \forall\, x \in A \setminus N, \tag{1A.18}$$

where $m^*(N) = 0$, hence

$$f_x \in \mathcal{R}(B), \quad \forall\, x \in A \setminus N. \tag{1A.19}$$

We present some examples that illustrate how Theorem 1A.2 contains results that generalize Theorem 3.1.9.

Example 1. Let $u \in \mathcal{R}(A \times B)$, take

$$\Omega \subset A \times B, \text{ contented}, \tag{1A.20}$$

and, for $x \in A$, set

$$\Omega_x = \{y \in B : (x, y) \in \Omega\}. \tag{1A.21}$$

Then, by Proposition 3.1.17,

$$f = \chi_\Omega u \in \mathcal{R}(A \times B). \tag{1A.22}$$

Note that, for $x \in A,\ y \in B$,

$$f_x(y) = \chi_{\Omega_x}(y) u_x(y). \tag{1A.23}$$

We form $g^\pm(x)$ as in (1A.11), and conclude from (1A.12) that

$$\int\limits_\Omega u\, dV = \int\limits_{A\times B} f\, dV = \int\limits_A g^\pm(x)\, dx. \tag{1A.24}$$

Let us specialize further, and assume

$$u \in C(A \times B), \quad \text{each } \Omega_x \subset B \text{ contented}. \tag{1A.25}$$

Then

$$g^\pm(x) = \int\limits_{\Omega_x} u(x, y)\, dy, \tag{1A.26}$$

and we have the following.

Proposition 1A.3. *Assume $\Omega \subset A \times B$ is contented and that u and Ω satisfy (1A.25). Then*

$$\int_{\Omega} u\, dV = \int_{A} \Big(\int_{\Omega_x} u(x,y)\, dy \Big) dx. \tag{1A.27}$$

(Sub)example 2. Take $B = [b_1, b_2]$ and

$$\Omega = \{(x,y) \in A \times B : \varphi_1(x) \le y \le \varphi_2(x)\}, \tag{1A.28}$$

where

$$\varphi_j : A \to [b_1, b_2], \quad \varphi_j \in \mathcal{R}(A), \quad \varphi_1 \le \varphi_2. \tag{1A.29}$$

It follows that Ω is contented, and also, for each $x \in A$,

$$\Omega_x = [\varphi_1(x), \varphi_2(x)] \tag{1A.30}$$

is contented. Take

$$u \in C(\overline{\Omega}), \quad f = \chi_\Omega u. \tag{1A.31}$$

Then, for each $x \in A$,

$$g^+(x) = g^-(x) = \int_{\varphi_1(x)}^{\varphi_2(x)} u(x,y)\, dy, \tag{1A.32}$$

and we deduce from Proposition 1A.3 that

$$\int_{\Omega} u\, dV = \int_{A} g^{\pm}(x)\, dx = \int_{A} \Big(\int_{\varphi_1(x)}^{\varphi_2(x)} u(x,y)\, dy \Big) dx. \tag{1A.33}$$

This generalizes Theorem 3.1.9.

Indeed, if $\Sigma \subset A$ is contented and

$$\varphi_1(x) = \varphi_2(x) \ \text{ for } \ x \in A \setminus \Sigma, \tag{1A.34}$$

then we can replace Ω by

$$\widetilde{\Omega} = \{(x,y) \in \Sigma \times B : \varphi_1(x) \le y \le \varphi_2(x)\}, \tag{1A.35}$$

and obtain

$$\int_{\widetilde{\Omega}} u\, dV = \int_{\Sigma} \Big(\int_{\varphi_1(x)}^{\varphi_2(x)} u(x,y)\, dy \Big) dx. \tag{1A.36}$$

Taking $u \equiv 1$, we have

$$V(\widetilde{\Omega}) = \int_{\Sigma} [\varphi_2(x) - \varphi_1(x)]\, dx. \tag{1A.37}$$

Volumes of balls. Taking $\widetilde{\Omega}$ to be the unit ball in $\mathbb{R}^n$,

$$B^n = \{z \in \mathbb{R}^n : |z| \le 1\}, \tag{1A.38}$$

we can apply (1A.37) to write

$$V(B^n) = 2 \int_{B^{n-1}} \sqrt{1 - |x|^2}\, dx. \tag{1A.39}$$

For example,

$$V(B^3) = 2 \int_{D} \sqrt{1 - |x|^2}\, dx, \tag{1A.40}$$

where $D = B^2$ is te unit disk. In turn, an application of Proposition 1A.3 gives

$$\int_{D} \sqrt{1 - |x|^2}\, dx = \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \sqrt{1 - x^2 - y^2}\, dy\, dx. \tag{1A.41}$$

Taking $a^2 = 1 - x^2$, we write the inner integral as

$$\begin{aligned} \int_{-a}^{a} \sqrt{a^2 - y^2}\, dy &= a^2 \int_{-1}^{1} \sqrt{1 - s^2}\, ds \\ &= \frac{\pi}{2} a^2, \end{aligned} \tag{1A.42}$$

using $y = as$ and the substitution $s = \sin t$. Hence

$$\int_{D} \sqrt{1 - |x|^2}\, dx = \frac{\pi}{2} \int_{-1}^{1} (1 - x^2)\, dx = \frac{2}{3}\pi, \tag{1A.43}$$

and we get

$$V(B^3) = \frac{4}{3}\pi. \tag{1A.44}$$

Another attack on the integral (1A.40), using polar coordinates, is mentioned in Exercise 6 of §3.1.

Another approach to computing $V(B^n)$ will arise from the following generalization of Theorem 3.1.9.

Proposition 1A.4. *Let $n = k + \ell$, and let $\Sigma \subset \mathbb{R}^k$ be a closed, bounded, contented set. Let $\varphi_j : \Sigma \to [0, \infty)$ be continuous, and satisfy $\varphi_1(x) \le \varphi_2(x)$. Take*

$$\Omega = \{(x, y) \in \mathbb{R}^n : x \in \Sigma, y \in \mathbb{R}^\ell,\ \varphi_1(x) \le |y| \le \varphi_2(x)\}. \tag{1A.45}$$

Then Ω is a contented set in $\mathbb{R}^n$. If $f : \Omega \to \mathbb{R}$ is continuous, then

$$g(x) = \int\limits_{\varphi_1(x) \le |y| \le \varphi_2(x)} f(x, y)\, dy \tag{1A.46}$$

is continuous on Σ, and

$$\int\limits_\Omega f\, dV_n = \int\limits_\Sigma g\, dV_k. \tag{1A.47}$$

Methods used to prove Theorem 3.1.9 can be tweaked to cover this result. Alternatively (and better), this proposition follows from Proposition 1A.3 in the same way as (1A.36) does.

Solids of revolution. Before applying Proposition 1A.4 to $V(B^n)$, we look at a class of 3D domains to which it applies, namely solids of revolution. Take a continuous function $\varphi : [a, b] \to [0, \infty)$, and consider

$$\Omega = \{(x, y, z) : a \le x \le b,\ \sqrt{y^2 + z^2} \le \varphi(x)\}. \tag{1A.48}$$

This has the form (1A.45), with $\Sigma = [a, b]$, $\varphi_1 \equiv 0$, $\varphi_2(x) = \varphi(x)$. If $f : \Omega \to \mathbb{R}$ is continuous, then (1A.46) leads to

$$g(x) = \int\limits_{|y| \le \varphi(x)} f(x, y)\, dy. \tag{1A.49}$$

In particular, if $f = f(x)$, then $g(x) = f(x) A(D_{\varphi(x)})$, with

$$D_\rho = \{y \in \mathbb{R}^2 : |y| \le \rho\}, \quad A(D_\rho) = \pi \rho^2, \tag{1A.50}$$

so, for Ω as in (1A.48),

$$\int\limits_\Omega f(x)\, dx\, dy\, dz = \pi \int_a^b f(x) \varphi(x)^2\, dx, \tag{1A.51}$$

and taking $f \equiv 1$ gives

$$V(\Omega) = \pi \int_a^b \varphi(x)^2\, dx. \tag{1A.52}$$

The ball B^3 is the solid of revolution one gets with $\varphi(x) = \sqrt{1 - x^2}$, $[a, b] = [-1, 1]$, so (1A.52) yields an alternative derivation of (1A.44):

$$V(B^3) = \pi \int_{-1}^1 (1 - x^2)\, dx = \frac{4}{3}\pi. \tag{1A.53}$$

Volumes of balls (bis). Returning to the case B^n, we apply Proposition 1A.4, with $\Sigma = [-1,1]$, $\varphi_1 \equiv 0$, $\varphi_2(x) = \sqrt{1-x^2}$, to obtain, for $f \in C(B^n)$,

$$\int\limits_{B^n} f\,dV = \int_{-1}^{1} \Bigl(\int\limits_{|y|\le\sqrt{1-x^2}} f(x,y)\,dy \Bigr) dx. \tag{1A.54}$$

In particular,

$$V(B^n) = \int_{-1}^{1} V(B^{n-1}_{\sqrt{1-x^2}})\,dx, \tag{1A.55}$$

where

$$B^{n-1}_r = \{y \in \mathbb{R}^{n-1} : |y| \le r\}. \tag{1A.56}$$

Scaling gives

$$V(B^{n-1}_r) = V(B^{n-1}) r^{n-1}, \tag{1A.57}$$

so we have the inductive result

$$V(B^n) = \beta_n V(B^{n-1}), \quad \beta_n = \int_{-1}^{1} (1-x^2)^{(n-1)/2}\,dx. \tag{1A.58}$$

Applying this to $n = 3$, and using $V(B^2) = A(D) = \pi$, leads back to (1A.53). To go one step further, we have

$$V(B^4) = \beta_4 V(B^3), \tag{1A.59}$$

with

$$\beta_4 = \int_{-1}^{1} (1-x^2)^{3/2}\,dx = 2\int_0^{\pi/2} \cos^4 t\,dt. \tag{1A.60}$$

One can attack this trigonometric integral using

$$2\cos^2 t = 1 + \cos 2t, \tag{1A.61}$$

and squaring it.

In §3.2 we show how to calculate the area of the unit sphere S^{n-1}, and relate this to a computation of $V(B^n)$.

§3.1B. Contented partitions and extension of Darboux's theorem

A contented partition of a cell $R \subset \mathbb{R}^n$ is a finite collection $\{K_j : 1 \le j \le M\}$ of compact subsets of R that are contented (so $\mathrm{cont}^+(K_j) = \mathrm{cont}^-(K_j) = V(K_j)$), satisfying

$$\bigcup_{j=1}^{M} K_j = R, \quad \mathrm{cont}(K_j \cap K_\ell) = 0, \quad \forall\, j \neq \ell. \tag{1B.1}$$

We denote the family of contented partitions of R by $C(R)$. This is larger than the family $\Pi(R)$ of cellular partitions, introduced in (3.1.8).

If $f : R \to \mathbb{R}$ is bounded and $\mathcal{P} = \{K_j : 1 \le j \le M\}$ is an element of $C(R)$, we define $\overline{I}_{\mathcal{P}}(f)$ and $\underline{I}_{\mathcal{P}}(f)$ as in (3.1.5):

$$\overline{I}_{\mathcal{P}}(f) = \sum_j (\sup_{K_j} f) V(K_j), \quad \underline{I}_{\mathcal{P}}(f) = \sum_j (\inf_{K_j} f) V(K_j). \tag{1B.2}$$

Alternatively, if we set

$$f^{\#}_{\mathcal{P}}(x) = \sup_{K_j} f, \quad f^{b}_{\mathcal{P}}(x) = \inf_{K_j} f, \quad \text{for } \ x \in K_j, \tag{1B.3}$$

with the convention that if x belongs to several sets K_ℓ, $f^{\#}_{\mathcal{P}}(x)$ is the maximum of such values and $f^{b}_{\mathcal{P}}(x)$ is the minimum, then $f^{\#}_{\mathcal{P}}, f^{b}_{\mathcal{P}} \in \mathcal{R}(R)$, and

$$\overline{I}_{\mathcal{P}}(f) = I(f^{\#}_{\mathcal{P}}), \quad \underline{I}_{\mathcal{P}}(f) = I(f^{b}_{\mathcal{P}}). \tag{1B.4}$$

Note that

$$f^{b}_{\mathcal{P}} \le f \le f^{\#}_{\mathcal{P}}, \tag{1B.5}$$

so, as in the case of the cellular partitions in (3.1.5),

$$\underline{I}_{\mathcal{P}}(f) \le \underline{I}(f) \le \overline{I}(f) \le \overline{I}_{\mathcal{P}}(f), \tag{1B.6}$$

for each $\mathcal{P} \in C(R)$ (with $\underline{I}(f)$ and $\overline{I}(f)$ still defined by (3.1.8)). In light of this, we can complement (3.1.8) with the identities

$$\overline{I}(f) = \inf_{\mathcal{P} \in C(R)} \overline{I}_{\mathcal{P}}(f), \quad \underline{I}(f) = \sup_{\mathcal{P} \in C(R)} \underline{I}_{\mathcal{P}}(f). \tag{1B.7}$$

Parallel to (3.1.3), we set

$$\mathrm{maxsize}(\mathcal{P}) = \max_j \mathrm{diam}\, K_j, \tag{1B.8}$$

for $\mathcal{P} \in C(R)$. Our next goal is to establish the following version of Darboux's theorem. Compare Theorem 1.1.4. See Exercise 30 of §3.2 for an extension to integrals on a C^1 surface $M \subset \mathbb{R}^n$.

Proposition 1B.1. *Let* $\mathcal{P}_\nu = \{K_{\nu j} : 1 \le j \le \nu\}$ *be a sequence of contented partitions of* R*, satisfying*

$$\text{maxsize}(\mathcal{P}_\nu) \le \delta_\nu \to 0. \tag{1B.9}$$

Let $f : R \to \mathbb{R}$ *be bounded. Then*

$$\overline{I}_{\mathcal{P}_\nu}(f) \longrightarrow \overline{I}(f), \quad \textit{and} \quad \underline{I}_{\mathcal{P}_\nu}(f) \longrightarrow \underline{I}(f). \tag{1B.10}$$

Consequently,

$$f \in \mathcal{R}(R) \Longleftrightarrow \overline{I}(f) = \lim_{\nu\to\infty} \sum_{j=1}^{\nu} f(\xi_{\nu j}) V(K_{\nu j}), \tag{1B.11}$$

for arbitrary $\xi_{\nu j} \in K_{\nu j}$*, in which case the limit is* $\int_R f\, dV$*.*

Proof. We make use of Proposition 3.1.11, which implies that

$$\begin{aligned} \overline{I}(f) &= \inf \Bigl\{ \int_R g\, dV : g \in C(R),\ g \ge f \Bigr\}, \\ \underline{I}(f) &= \sup \Bigl\{ \int_R g\, dV : g \in C(R),\ g \le f \Bigr\}. \end{aligned} \tag{1B.12}$$

We will concentrate on proving the first part of (1B.10), i.e.,

$$\overline{I}_{\mathcal{P}_\nu}(f) \longrightarrow \overline{I}(f). \tag{1B.13}$$

The second part has a similar proof.

To begin, pick $\varepsilon > 0$, and take $\gamma > 0,\ g_1 \in C(R)$ such that

$$f + \gamma \le g_1, \quad I(g_1) \le \overline{I}(f) + \varepsilon. \tag{1B.14}$$

Now pick ν sufficiently large that, for $x, x' \in R$,

$$|x - x'| \le 3\delta_\nu \Longrightarrow |g_1(x) - g_1(x')| < \frac{\gamma}{2}. \tag{1B.15}$$

Then, for all $x \in R$,

$$f^{\#}_{\mathcal{P}_\nu}(x) \le \sup_{|x'-x| \le 3\delta_\nu} f(x') \le \sup_{|x'-x| \le 3\delta_\nu} g_1(x') - \gamma \le g_1(x), \tag{1B.16}$$

so

$$\overline{I}_{\mathcal{P}_\nu}(f) = I(f^{\#}_{\mathcal{P}_\nu}) \le I(g_1) \le \overline{I}(f) + \varepsilon. \tag{1B.17}$$

Since $\overline{I}(f) \le \overline{I}_{\mathcal{P}_\nu}(f)$, this leads to (1B.13).

§3.1C. From coverings to partitions

Our goal here is to prove the following.

Proposition 1C.1. *Let $R \subset \mathbb{R}^n$ be a cell. Assume $\{U_k : 1 \le k \le K\}$ is a finite cover of R by open cells. Then there is a partition $\mathcal{P} = \{L_\alpha\}$ of R with the property that*

$$\text{each } L_\alpha \text{ is contained in some } U_k. \tag{1}$$

In this work, a cell R is defined to be a product of compact intervals. By contrast, we say an *open cell* (in R) is a product of intervals that is open (in R).

To start the proof of Proposition 1C.1, we note that scaling each x_j variable leaves invariant the class of cells, open cells, coverings, and partitions, so without loss of generality we can assume

$$R = [0,1] \times \cdots \times [0,1]. \tag{2}$$

Generally, if a cell U (open or closed) is a product of intervals with endpoints a_k and b_k, $1 \le k \le n$, we call the points $(c_1, \dots, c_n) \in R$ whose kth component is either a_k or b_k the *vertices* of U. Returning to the cover $\{U_k\}$, we can write each U_k as a union of open cells (in R), $U_{k\ell}, \ell \in \mathbb{N}$, with rational vertices. Hence we get a cover of R by a countable family $\{U_{k\ell} : 1 \le k \le K, \ell \in \mathbb{N}\}$ of open cells with rational vertices. Since R is compact, this has a finite subcover,

$$\{V_j : 1 \le j \le J\}, \tag{3}$$

of open cells with rational vertices. It suffices to show there is a partition $\mathcal{P} = \{L_\alpha\}$ such that

$$\text{each } L_\alpha \text{ is contained in some } V_j. \tag{4}$$

To arrange this, let μ be the least common multiple of the denominators of the rational numbers that arise as components of the vertices of the various V_j. Then take for $\mathcal{P}$ the partition of $[0,1]^n$ into μ^n cubes, with vertices having components

$$\frac{\alpha}{\mu}, \quad \alpha \in \mathbb{Z}, \quad 0 \le \alpha \le \mu. \tag{5}$$

This partition has the property (4).

Remark. Proposition 1C.1 provides a convenient means of passing from the covering $\{R_1, \dots, R_N, R_1^\#, \dots, R_M^\#\}$ to the partition $\mathcal{P} = \{L_k\}$ in the proof of Proposition 3.1.31.

§3.2A. Action of diffeomorphisms on surfaces

Let $\Omega, \mathcal{O} \subset \mathbb{R}^n$ be open, and suppose $F : \Omega \to \mathcal{O}$ is a C^k diffeomorphism. Let $S \subset \Omega$ be a C^k surface, with coordinate chart

$$\varphi : U \longrightarrow S, \quad U \subset \mathbb{R}^\ell, \text{ open,} \tag{1}$$

so

$$x \in U \Longrightarrow D\varphi(x) : \mathbb{R}^\ell \to \mathbb{R}^n \text{ is injective.} \tag{2}$$

We aim to prove the folowing.

Proposition 3.2A.1. *Let $\Sigma = F(S)$. Then $\Sigma \subset \mathcal{O}$ is a C^k surface, with coordinate chart $\psi = F \circ \varphi$. In addition,*

$$p \in S,\ q = F(p) \in \Sigma \Longrightarrow DF(p) : T_pS \xrightarrow{\approx} T_q\Sigma. \tag{3}$$

Proof. The chain rule gives

$$D\psi(x) = DF(\varphi(x))D\varphi(x), \quad x \in \Omega. \tag{4}$$

Hence $D\psi(x)$ is injective for each $x \in U$, so ψ is a coordinate chart on Σ, making Σ a C^k surface.

Say $\varphi(x_0) = p$, so $\psi(x_0) = q$. Then $T_pS = \operatorname{Range} D\varphi(x_0)$, and

$$T_q\Sigma = \operatorname{Range} D\psi(x_0) = \operatorname{Range} DF(p)D\varphi(x_0) = DF(p)T_pS, \tag{5}$$

and we have (3).

We next compare the metric tensors on S and Σ (in φ and ψ coordinates, respectively). We have, respectively,

$$G(x) = D\varphi(x)^t D\varphi(x), \tag{6}$$

and

$$\begin{aligned} H(x) &= D\psi(x)^t D\psi(x) \\ &= D\varphi(x)^t [DF(\varphi(x))^t DF(\varphi(x))] D\varphi(x). \end{aligned} \tag{7}$$

This leads to the following.

Proposition 3.2A.2. *If $F : \Omega \to \mathcal{O}$ is an isometry, given by $F(y) = Ay,\ A \in O(n)$, then, for $x \in U$,*

$$H(x) = G(x). \tag{8}$$

§3.2B. Smooth maps between surfaces

Let $X \subset \mathbb{R}^m$ and $Y \subset \mathbb{R}^n$ be surfaces, smooth of class C^k. Say $j = \dim X$, $\ell = \dim Y$. We say a map $F : X \to Y$ is smooth of class C^k provided that, for each $p \in X$, there are coordinate charts $\varphi : \Omega \to U \ni p$ and $\psi : \mathcal{O} \to V \ni q = F(p)$ such that

$$\psi^{-1} \circ F \circ \varphi : \Omega \longrightarrow \mathcal{O} \quad \text{is smooth of class } C^k. \tag{1}$$

Note that Lemma 3.2.1 then gives such a result for other coordinate charts about p and q.

The concept of smoothness of maps between surfaces is applicable in particular to the case where one surface is actually an open set in Euclidean space. In connection with this, it is useful to note that the arguments proving Lemma 3.2.1 also establish the following.

Lemma 3.2B.1. *Let $Y \subset \mathbb{R}^n$ be a C^k smooth surface, $q \in Y$. Then there exist a neighborhood $\widetilde{V}$ of q in $\mathbb{R}^n$ and a C^k map*

$$R : \widetilde{V} \longrightarrow Y \cap \widetilde{V}, \quad R(y) = y, \ \forall\, y \in Y \cap \widetilde{V}. \tag{2}$$

Proof. Say $\psi(y_0) = q$. As seen in the proof of Lemma 3.2.1, the inverse function theorem implies there exist a neighborhood $\widetilde{\mathcal{O}} = \mathcal{O}_1 \times B$ of $(y_0, 0)$ in $\mathcal{O} \times \mathbb{R}^{n-\ell}$, a neighborhood $\widetilde{V}$ of q in $\mathbb{R}^n$, and a diffeomorphism

$$\Psi : \widetilde{\mathcal{O}} \longrightarrow \widetilde{V}, \quad \text{such that} \ \ \Psi(y, 0) = \psi(y), \quad \forall\, y \in \mathcal{O}_1. \tag{3}$$

Then take

$$R = \Psi \circ P \circ \Psi^{-1}, \quad P(y, v) = (y, 0). \tag{4}$$

Using this, we can prove the following.

Proposition 3.2B.2. *Take X, Y as above, and let $F : X \to Y$. Then F is C^k smooth as a map from X to Y if and only if $F : X \to \mathbb{R}^n$ is C^k smooth.*

Proof. Take $p \in X$, $q = F(p) \in Y$, and then take φ, ψ, U, V as in the definition above. If $F : X \to Y$ is C^k smooth, then

$$F \circ \varphi = \psi \circ (\psi^{-1} \circ F \circ \varphi) : \Omega \longrightarrow \mathbb{R}^n \quad \text{is } C^k \text{ smooth}, \tag{5}$$

so $F : X \to \mathbb{R}^n$ is smooth. Conversely, if $F : X \to \mathbb{R}^n$ is C^k smooth (and $F : X \to Y$), take $\widetilde{V}$ and R as in Lemma 3.2B.1. We have

$$F = R \circ F, \tag{6}$$

for x in some neighborhood of p in X, so $F : X \to Y$ is C^k smooth, in a neighborhood of p.

If we denote the composite map (1) by $G = \psi^{-1} \circ F \circ \varphi : \Omega \to \mathcal{O}$, which by hypothesis is C^k-smooth, we have

$$F \circ \varphi = \psi \circ G : \Omega \to Y. \tag{7}$$

If $\varphi(x_0) = p$ and $\psi(y_0) = q$, we have

$$D\varphi(x_0) : \mathbb{R}^j \xrightarrow{\approx} T_pX, \quad D\psi(y_0) : \mathbb{R}^\ell \xrightarrow{\approx} T_qY, \tag{8}$$

and hence it is natural to define

$$DF(p) : T_pX \longrightarrow T_qY \tag{9}$$

as

$$DF(p) = D\psi(y_0)\, DG(x_0)\, D\varphi(x_0)^{-1}. \tag{10}$$

Note that $DF(p)$ is invertible if and only if $DG(x_0)$ is. In such a case, the inverse function theorem for G translates to the following result for F:

Proposition 3.2B.3. *Let $F : X \to Y$ be a C^k-smooth map, $k \geq 1$, $p \in X$, $q = F(p)$. If $DF(p) : T_pX \to T_qY$ is an isomorphism, then there exist a neighborhood U of p in X and a neighborhood V of q in Y such that $F : U \to V$ is one-to-one and onto. Furthermore, the inverse map $F^{-1} : V \to U$ is smooth of class C^k.*

Proof. For the last result, note that

$$G^{-1} = \varphi^{-1} \circ F^{-1} \circ \psi \tag{11}$$

is C^k-smooth, so (1) applies to F^{-1}.

§3.2C. Further surface integrals

We have the goal of evaluating

$$S_n(\alpha) = \int\limits_{S^{n-1}} \omega^\alpha \, dS(\omega), \quad \alpha = (\alpha_1, \dots, \alpha_n). \tag{1}$$

We accomplish this by evaluating

$$I_n(\alpha) = \int\limits_{\mathbb{R}^n} e^{-|x|^2} x^\alpha \, dx \tag{2}$$

in two ways. For one, we write this integral as an iterated integral and use the identity $e^{-|x|^2} = e^{-x_1^2} \cdots e^{-x_n^2}$ to get

$$I_n(\alpha) = I_1(\alpha_1) \cdots I_1(\alpha_n), \tag{3}$$

where, for $k \in \mathbb{Z}^+$,

$$I_1(k) = \int\limits_{\mathbb{R}} e^{-x^2} x^k \, dx. \tag{4}$$

We note right away that k odd $\Rightarrow I_1(k) = 0$, hence

$$I_n(\alpha) = 0, \quad \text{if any component } \alpha_\nu \text{ is odd.} \tag{5}$$

For the second evaluation of (2), we use (3.2.28) to write

$$\begin{aligned} I_n(\alpha) &= \int\limits_{S^{n-1}} \int_0^\infty e^{-r^2} r^{|\alpha|} \omega^\alpha r^{n-1} \, dr \, dS(\omega) \\ &= S_n(\alpha) \int_0^\infty e^{-r^2} r^{n+|\alpha|-1} \, dr. \end{aligned} \tag{6}$$

Taking $s = r^2$ gives

$$\begin{aligned} \int_0^\infty e^{-r^2} r^{n+|\alpha|-1} \, dr &= \frac{1}{2} \int_0^\infty e^{-s} s^{(n+|\alpha|)/2-1} \, ds \\ &= \frac{1}{2} \Gamma\Big(\frac{n+|\alpha|}{2}\Big), \end{aligned} \tag{7}$$

where, as in (3.2.33), we have the gamma function

$$\Gamma(z) = \int_0^\infty e^{-s} s^{z-1} \, ds, \quad z > 0. \tag{8}$$

Note also that, for $k \in \mathbb{Z}^+$,

$$I_1(2k) = 2\int_0^\infty e^{-x^2} x^{2k}\,dx = \Gamma\Big(k + \frac{1}{2}\Big). \tag{9}$$

We have the following conclusions:

$$S_n(\alpha) = 0, \quad \text{whenever some component } \alpha_\nu \text{ is odd,} \tag{10}$$

and, for $\beta \in (\mathbb{Z}^+)^n$,

$$\begin{aligned} \int\limits_{S^{n-1}} \omega^{2\beta}\,dS(\omega) &= 2\frac{I_n(2\beta)}{\Gamma(|\beta| + \frac{n}{2})} \\ &= 2\frac{\Gamma(\beta_1 + \frac{1}{2})\cdots\Gamma(\beta_n + \frac{1}{2})}{\Gamma(|\beta| + \frac{n}{2})}. \end{aligned} \tag{11}$$

In connection with this, recall that, for $k \in \mathbb{Z}^+$,

$$\Gamma(k+1) = k! \tag{12}$$

and

$$\Gamma(k + \tfrac{1}{2}) = (k - \tfrac{1}{2})(k - \tfrac{3}{2})\cdots\tfrac{1}{2}\sqrt{\pi}. \tag{13}$$

As an aside, we derive a formula for the integral of x^α over the unit ball $B^n \subset \mathbb{R}^n$:

$$\begin{aligned} \int\limits_{B^n} x^\alpha\,dx &= \int\limits_{S^{n-1}} \int_0^1 \omega^\alpha r^{n+|\alpha|-1}\,dr\,dS(\omega) \\ &= S_n(\alpha)\int_0^1 r^{n+|\alpha|-1}\,dr \\ &= \frac{S_n(\alpha)}{n + |\alpha|}. \end{aligned} \tag{14}$$

§3.4. Sard's theorem.

The following result, which complements Proposition 3.4.1, is the more typical formulation of Sard's theorem.

Proposition 3.4.1. *If $\mathcal{O} \subset \mathbb{R}^n$ is open and $F : \mathcal{O} \to \mathbb{R}^n$ is a C^1 map, C its set of critical points, then*

$$m^*(F(C)) = 0. \tag{1}$$

Proof. Take compact $K_\ell \subset \mathcal{O}$ such that $K_\ell \nearrow \mathcal{O}$. Set $S_\ell = F(C \cap K_\ell), S = F(C)$. Then each S_ℓ is compact, and $m^*(S_\ell) = \text{cont}^+(S_\ell) = 0$, while $S_\ell \nearrow S$. It is a general result that

$$S_\ell \nearrow S,\ m^*(S_\ell) = 0 \Longrightarrow m^*(S) = 0. \tag{2}$$

To see this, pick $\varepsilon > 0$ and let $\{U_{\ell\nu} : \nu \in \mathbb{N}\}$ be a countable cover of S_ℓ by cells such that $\sum_\nu V(U_{\ell\nu}) < 2^{-\ell}\varepsilon$. Then $\{U_{\ell\nu} : \ell, \nu \in \mathbb{N}\}$ is a countable cover of S, and $\sum_{\ell,\nu} V(U_{\ell\nu}) < 2\varepsilon$. Hence $m^*(S) < 2\varepsilon$ for each $\varepsilon > 0$. This proves (1).

Note that the analogue of (2) fails when m^* is replaced by cont^+, as is illustrated by $S_\ell \nearrow [0,1] \cap \mathbb{Q}$, where S_ℓ is a set with ℓ points.

§3.5A. Gradient vector fields on a surface

Let $S \subset \mathbb{R}^n$ be a k-dimensional surface, smooth of class C^ℓ. Assume $\mathcal{O}$ is an open neighborhood of S in $\mathbb{R}^n$ and $f : \mathcal{O} \to \mathbb{R}$ is a C^ℓ-smooth function. Hence ∇f is a vector field on $\mathcal{O}$. We associate to this a vector field tangent to S via the formula

$$\nabla^S f(y) = P(y)\nabla f(y), \quad y \in S, \tag{1}$$

where

$$P(y) : \mathbb{R}^n \longrightarrow T_y S \ \text{ is the orthogonal projection.} \tag{2}$$

We want some further formulas for $\nabla^S f$, in terms of a coordinate chart

$$\varphi : \Omega \longrightarrow U \subset S, \quad \Omega \subset \mathbb{R}^k \ \text{ open,} \tag{3}$$

where φ is C^ℓ-smooth and $D\varphi(x) : \mathbb{R}^k \to \mathbb{R}^n$ is injective for each $x \in \Omega$. In particular, we want to relate $\nabla^S f$ to ∇g, where

$$g = f \circ \varphi : \Omega \longrightarrow \mathbb{R}. \tag{4}$$

To begin, the chain rule gives

$$Dg(x) = Df(y)D\varphi(x), \quad y = \varphi(x), \tag{5}$$

so, for $Y \in \mathbb{R}^k$,

$$\begin{aligned} \nabla g(x) \cdot Y &= Dg(x)Y \\ &= Df(y)D\varphi(x)Y \\ &= \nabla f(y) \cdot D\varphi(x)Y \\ &= D\varphi(x)^t \nabla f(y) \cdot Y. \end{aligned} \tag{6}$$

We hence have

$$\begin{aligned} \nabla g(x) &= D\varphi(x)^t \nabla f(y) \\ &= D\varphi(x)^t \nabla^S f(y), \end{aligned} \tag{7}$$

the last identity because

$$\operatorname{Ker} D\varphi(x)^t = (\operatorname{Range} D\varphi(x))^\perp = (T_y S)^\perp = \operatorname{Ker} P(y). \tag{8}$$

Note that

$$D\varphi(x)^t : T_y S \xrightarrow{\approx} \mathbb{R}^k, \quad D\varphi(x) : \mathbb{R}^k \xrightarrow{\approx} T_y S. \tag{9}$$

We have seen the "metric tensor"

$$G(x) = D\varphi(x)^t D\varphi(x) \in \mathcal{L}(\mathbb{R}^k), \tag{10}$$

which is symmetric and positive definite. To relate $\nabla^S f(y)$ to a vector field on Ω, we want to solve

$$D\varphi(x)X = \nabla^S f(y), \quad \text{for} \quad X \in \mathbb{R}^k. \tag{11}$$

Indeed, applying $D\varphi(x)^t$ to both sides yields

$$G(x)X = D\varphi(x)^t \nabla^S f(y), \tag{12}$$

hence

$$X = G(x)^{-1} D\varphi(x)^t \nabla^S f(y). \tag{13}$$

Comparison with (7) gives the formula

$$X = G(x)^{-1} \nabla g(x), \tag{14}$$

or, denoting the solution X by $\nabla^S g(x)$,

$$\nabla^S g(x) = G(x)^{-1} \nabla g(x). \tag{15}$$

This is the representative of the vector field $\nabla^S f$ on S, in the coordinate system (3).

In turn, (11) and (15) yield

$$\nabla^S f(y) = D\varphi(x) G(x)^{-1} \nabla g(x), \tag{16}$$

hence, by (7),

$$\nabla^S f(y) = D\varphi(x) G(x)^{-1} D\varphi(x)^t \nabla f(y). \tag{17}$$

That is to say, we have the formula

$$P(y) = D\varphi(x) G(x)^{-1} D\varphi(x)^t, \quad y = \varphi(x), \tag{18}$$

for the orthogonal projection $P(y)$ defined in (2). In connection with this, we mention that the form of the right side of (18) readily yields

$$P(y)^t = P(y), \quad \text{and} \quad P(y)^2 = P(y). \tag{19}$$

Remark. The orthogonal projection P will play an important role in geometric investigations of surfaces, pursued in Chapter 6.

Chapter 4. Differential forms and the Gauss-Green-Stokes formula

§4.1. Revision of (4.1.12)–(4.1.13)

There is a special notation we use for k-forms. If $J = (j_1, \dots, j_k)$ and $1 \le j_1 < \cdots < j_k \le n$, we set

$$\alpha = \sum_J a_J(x)\, dx_{j_1} \wedge \cdots \wedge dx_{j_k}, \tag{4.1.12}$$

where

$$a_J(x) = \alpha(D_{j_1}, \dots, D_{j_k}), \quad D_j = \partial/\partial x_j. \tag{4.1.13}$$

More generally, we assign meaning to (4.1.12) summed over all k-indices $(j_1, \dots, j_k)$, where we identify

$$dx_{j_1} \wedge \cdots \wedge dx_{j_k} = (\operatorname{sgn} \sigma)\, dx_{j_{\sigma(1)}} \wedge \cdots \wedge dx_{j_{\sigma(k)}}, \tag{4.1.14}$$

σ being a permutation of $\{1, \dots, k\}$. In such a case, we replace (4.1.13) by

$$a_J(x) = \frac{1}{k!} \alpha(D_{j_1}, \dots, D_{j_k}). \tag{4.1.13A}$$

Another variation on (4.1.12), parallel to the J-index notation used in (2.1.35) and (2.1.39), is

$$\alpha = \sum_J a_J(x)\, dx^J, \quad dx^J = dx_{j_1} \wedge \cdots \wedge dx_{j_k}. \tag{4.1.12A}$$

Then we can rewrite the formula (4.2.11) for the exterior derivative as

$$d\alpha = \sum_{J,\ell} \frac{\partial a_J}{\partial x_\ell}\, dx_\ell \wedge dx^J. \tag{4.2.11A}$$

§4.1A. k-forms on a surface

Let $S \subset \mathbb{R}^n$ be a C^ℓ-smooth m-dimensional surface. We have developed results on k-forms on an open neighborhood of S in $\mathbb{R}^n$. Here we introduce the notion of a k-form on S itself.

Say S is covered by U_j, for which there are smooth coordinate charts

$$\varphi_j : \mathcal{O}_j \longrightarrow U_j, \quad \mathcal{O}_j \subset \mathbb{R}^m, \text{ open.} \tag{1}$$

For a first approach, we view a k-form on S as an object associated to a collection of k-forms α_j on $\mathcal{O}_j$, subject to a natural compatibility condition on coordinate overlaps. Namely, if $U_i \cap U_j \neq \emptyset$, we have transition maps

$$F_{ij} : \mathcal{O}_{ij} \longrightarrow \mathcal{O}_{ji}, \quad F_{ij} = \varphi_j^{-1} \circ \varphi_i, \tag{2}$$

known to be C^ℓ-smooth, by Lemma 3.2.1. The compatibility condition is

$$\alpha_i = F_{ij}^* \alpha_j \ \text{ on } \ \mathcal{O}_{ij}. \tag{3}$$

We say the collection $\{\alpha_j\}$ of k-forms is *compatible*. Recalling that the formula for the pull-back F_{ij}^* involves first-order derivatives of the map F_{ij}, we see that if S is a C^ℓ-smooth surface, a natural smoothness condition for a compatible collection $\{\alpha_j\}$ is that these forms be C^r-smooth, with $r \leq \ell - 1$. We give the compatible collection $\{\alpha_j\}$ of k-forms a label, say α.

One way a compatible collection of k-forms arises is the following. Suppose $\mathfrak{U}$ is an open neighborhood of S in $\mathbb{R}^n$ and $\tilde{\alpha}$ is a C^r-smooth k-form on $\mathfrak{U}$. Then each φ_j is also a map $\varphi_j : \mathcal{O}_j \to \mathfrak{U}$, and we can set

$$\alpha_j = \varphi_j^* \tilde{\alpha} \in \Lambda^k(\mathcal{O}_j). \tag{4}$$

The compatibility condition (3) follows from (4.1.28), in this case. We say

$$\alpha = \iota^* \tilde{\alpha}, \tag{5}$$

where $\iota : S \hookrightarrow \mathfrak{U}$ is the inclusion, and α is the label for the compatible collection $\{\alpha_j\}$.

A bit later we will establish a converse to this construction.

Suppose now that we have a C^ℓ-smooth map

$$\psi : \Omega \longrightarrow S, \quad \Omega \subset \mathbb{R}^d, \ \text{ open,} \tag{6}$$

and a compatible collection $\alpha = \{\alpha_j\}$ of C^r-smooth k-forms on S (i.e., associated to a coordinate cover $\{\varphi_j\}$ of S). We propose to define

$$\psi^* \alpha \in \Lambda^k(\Omega). \tag{7}$$

To do this, we set

$$\Omega_j = \psi^{-1}(U_j), \quad \Omega = \bigcup_j \Omega_j, \tag{8}$$

with U_j as in (1), and define

$$\begin{aligned} &G_j : \Omega_j \longrightarrow \mathcal{O}_j, \quad G_j = \varphi_j^{-1} \circ \psi, \\ &\beta_j = G_j^* \alpha_j, \quad \beta_j \in \Lambda^k(\Omega_j). \end{aligned} \tag{9}$$

The compatibility condition on $\{\alpha_j\}$ implies

$$\beta_i = \beta_j \ \text{ on } \ \Omega_i \cap \Omega_j, \tag{10}$$

if $\Omega_j \cap \Omega_j \neq \emptyset$. Hence there is a unique k-form $\beta \in \Lambda^k(\Omega)$, equal to β_j on each open subset Ω_j. We set

$$\psi^* \alpha = \beta. \tag{11}$$

Note that taking $\psi = \varphi_j : \mathcal{O}_j \to S$ as in (1) yields

$$\varphi_j^* \alpha = \alpha_j. \tag{12}$$

Also, if α is given by (5), we have

$$\psi^* \alpha = \psi^* \tilde{\alpha}, \tag{13}$$

where, for the right side of (13), we take $\psi : \Omega \to \mathfrak{U}$.

Going further, suppose we have a C^ℓ-smooth, d-dimensional surface $X \subset \mathbb{R}^\nu$, covered by V_i, for which there are smooth coordinate charts

$$\psi_i : \Omega_i \longrightarrow V_i, \quad \Omega_i \subset \mathbb{R}^d, \text{ open}, \tag{14}$$

and suppose we have a C^ℓ-smooth map

$$F : X \longrightarrow S, \tag{15}$$

as considered in §3.2B of this supplement. Thus, for each i, we have a C^ℓ-smooth map

$$F_i = F \circ \psi_i : \Omega_i \longrightarrow S, \tag{16}$$

and, as established above, if $\alpha = \{\alpha_j\}$ is a compatible collection of k-forms on S, we have for each i a well defined k-form

$$\beta_i = F_i^* \alpha \in \Lambda^k(\Omega_i). \tag{17}$$

Furthermore, $\{\beta_i\}$ is a compatible collection of k-forms on X (associated to the coordinate cover $\{\psi_i\}$). We label this collection β, and obtain

$$\beta = F^*\alpha. \tag{18}$$

Parallel to (13), if α is given by (5), we have

$$F^*\alpha = F^*\tilde{\alpha}, \tag{19}$$

where, for the right side of (19), we take $F : X \to \mathfrak{U}$, and $F^*\tilde{\alpha}$ labels the compatible collection of k-forms $\{F_i^*\tilde{\alpha}\}$.

We record observations on exterior derivatives and wedge products. If $\alpha = \{\alpha_j\}$ and $\beta = \{\beta_j\}$ are compatible collections of k-forms and ℓ-forms on S (associated to a coordinate cover $\{\varphi_j\}$), then $\{d\alpha_j\}$ and $\{\alpha_j \wedge \beta_j\}$ are compatible collections, defining

$$d\alpha = \{d\alpha_j\}, \quad \alpha \wedge \beta = \{\alpha_j \wedge \beta_j\}. \tag{20}$$

Also we have

$$F^*d\alpha = dF^*\alpha, \quad F^*(\alpha \wedge \beta) = F^*\alpha \wedge F^*\beta, \tag{21}$$

for F as in (15).

We turn to a construction of compactly supported "k-forms" on S, in a fashion that interfaces with (5). Let $\varphi_0 : \mathcal{O}_0 \to U_0 \subset S$ be a C^ℓ-smooth coordinate chart on S, and let α_0 be a C^r-smooth k-form on $\mathcal{O}_0$ ($r \le \ell - 1$), with compact support. Then $\varphi_0(\operatorname{supp} \alpha_0) = K_0$ is a compact subset of U_0. We aim to prove the following.

Proposition 4.1A.1. *Let $\mathfrak{U}$ be an open neighborhood of K_0 in $\mathbb{R}^n$. Then there is a C^r-smooth k-form $\tilde{\alpha}$ on $\mathbb{R}^n$ such that* $\operatorname{supp} \tilde{\alpha} \subset \mathfrak{U}$ *and*

$$\varphi_0^*\tilde{\alpha} = \alpha_0. \tag{22}$$

Proof. Chopping up α_0 via a partition of unity and applying some relabeling, we can refer to the proof of Lemma 3.2B.1 of this supplement, and assume there exist an open neighborhood $\mathcal{O}_0'$ of $\operatorname{supp} \alpha_0$ in $\mathcal{O}_0$, an open neighborhood B of 0 in $\mathbb{R}^{n-m}$ ($m = \dim S$), and a diffeomorphism

$$\Phi : \mathcal{O}_0' \times B \longrightarrow \widetilde{\mathfrak{U}} \subset \mathfrak{U}, \ \text{ such that } \ \Phi(x, 0) = \varphi_0(x), \ \forall\, x \in \mathcal{O}_0'. \tag{23}$$

Denote elements of $\mathcal{O}_0' \times B$ by $(x, v),\ x \in \mathcal{O}_0',\ v \in B$. Now take

$$\beta_0 = fP^*\alpha_0, \quad P(x, v) = x, \tag{24}$$

where $f \in C_0^\infty(B)$ and $f(v) = 1$ for v in some neighborhood of 0. We have β_0 compactly supported in $\mathcal{O}_0' \times B$. Then set

$$\tilde{\alpha} = (\Phi^{-1})^*\beta_0, \quad \operatorname{supp} \tilde{\alpha} \subset \widetilde{\mathfrak{U}}, \tag{25}$$

to obtain (22).

Remark. Having Proposition 4.1A.1, we can apply (5), to obtain

$$\alpha = \iota^*\tilde{\alpha}, \tag{26}$$

supported on U_0, and satisfying $\varphi_0^*\alpha = \alpha_0$, in the sense of (7)–(12).

Another approach to k-forms on a surface.

Let $S \subset \mathbb{R}^n$ be a smooth m-dimensional surface, covered by U_j, for which there are coordinate charts,

$$\varphi_j : \mathcal{O}_j \longrightarrow U_j, \quad \mathcal{O}_j \subset \mathbb{R}^m, \quad \text{open.}$$

In §4.1A we have developed the notion of a k-form α on S as an object associated to a collection of k-forms α_j on $\mathcal{O}_j$, subject to natural compatibility conditions on coordinate overlaps. Here we outline another approach to the definition of a k-form on S.

To set up this definition, we bring in the space

$$T^k S = \{(x, v_1, \dots, v_k) \in (\mathbb{R}^n)^{k+1} : x \in S,\ v_j \in T_x S\},$$

which has the structure of a smooth $(k+1)m$-dimensional surface. Then we define a k-form α on S to be a smooth map

$$\alpha : T^k S \longrightarrow \mathbb{R},$$

having the property that, for each $x \in S$, the restriction

$$\alpha(x) : T_x S \times \cdots \times T_x S \longrightarrow \mathbb{R}$$

is k-linear, and alternating.

In this setting, we can directly define $\alpha_j \in \Lambda^k(\mathcal{O}_j)$ by the formula

$$\alpha_j(x)(X_1, \dots, X_k) = \alpha(\varphi_j(x))(D\varphi_j(x)X_1, \dots, D\varphi_j(x)X_k),$$

with

$$X_\nu \in \mathbb{R}^m, \quad D\varphi_j(x)X_\nu \in T_{\varphi_j(x)}S.$$

Going further, if $X \subset \mathbb{R}^d$ is a smooth surface and $F : X \to S$ is a smooth map (as defined in §3.2B of this Supplement), we have

$$DF(x) : T_x X \longrightarrow T_{F(x)}S,$$

defined as in (7)–(10) of §3.2B, and then, for $x \in X$, α as above,

$$F^*\alpha(x)(X_1, \dots, X_k) = \alpha(F(x))(DF(x)X_1, \dots, DF(x)X_k),$$

with

$$X_\nu \in T_x X, \quad DF(x)X_\nu \in T_{F(x)}S.$$

§4.2A. Exterior derivative versus gradient

As stated in (4.1.8), a 1-form α acts linearly on a vector field X by

$$\alpha(X)=\sum_j b^j(x)a_j(x),\quad X=\sum b^j(x)\frac{\partial}{\partial x_j},\ \alpha=\sum a_j(x)\,dx_j. \tag{1}$$

Say these objects are defined on Ω, an open subset of $\mathbb{R}^n$. If we have a Riemannian metric on Ω, given by a positive definite matrix $G(x)$, we can associate to α a vector field Y such that

$$\alpha(X)=G(x)X\cdot Y. \tag{2}$$

That is,

$$Y=\sum a^j(x)\frac{\partial}{\partial x_j},\quad \sum_k g_{jk}(x)a^k(x)=a_j(x), \tag{3}$$

where $(g_{jk})=G$. Equivalently,

$$a^j(x)=\sum_k g^{jk}(x)a_k(x),\quad (g^{jk}(x))=G(x)^{-1}. \tag{4}$$

One suggestive way to think of this is to regard

$$G(X):T_x\Omega\xrightarrow{\approx}T_x^*\Omega, \tag{5}$$

and then

$$\alpha(x)=G(x)Y(x),\ \text{ i.e., }\ Y(x)=G(x)^{-1}\alpha(x). \tag{6}$$

In case $\alpha=df(x)$, we define the gradient $\nabla^G f$ to be the associated vector field,

$$\nabla^G f(x)=G(x)^{-1}df(x), \tag{7}$$

the superscript G denoting that we are associating the vector field $\nabla^G f$ to the 1-form df via the metric tensor G. Compare the formula (15) in §3.5A of this supplement.

§4.5. Alternative endgame to the change of variable formula for integrals

Recall the change of variable formula:

Theorem 4.5.1. *Let $\mathcal{O}, \Omega$ be open sets in $\mathbb{R}^n$, $\varphi : \mathcal{O} \to \Omega$ a C^1 diffeomorphism. Given f continuous on Ω, with compact support, we have*

$$\int_{\mathcal{O}} f(\varphi(x)) |\det D\varphi(x)|\, dx = \int_{\Omega} f(x)\, dx. \tag{1}$$

This was deduced from the following.

Theorem 4.5.2. *Let $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ be a C^1 map. Assume $\varphi(x) = x$ for $|x| \geq R$. Let f be a continuous function on $\mathbb{R}^n$ with compact support. Then*

$$\int f(\varphi(x))\ \det D\varphi(x)\, dx = \int f(x)\, dx. \tag{2}$$

The proof of Theorem 4.5.2 given in the text was a neat variant of an approach of P. Lax. The derivation of Theorem 4.5.1 from 4.5.2 involved further arguments, and made use of the fact that $G\ell_+(n, \mathbb{R})$ is connected. Here we provide an alternative route from Theorem 4.5.2 to 4.5.1, making use of Proposition 3.1.10, which we recall.

Lemma A. *Let f be a continuous function with compact support on $\mathbb{R}^n$. If $A \in G\ell(n, \mathbb{R})$, then*

$$\int f(x)\, dx = |\det A| \int f(Ax)\, dx. \tag{3}$$

To start the proof of Theorem 4.5.1, bring in a partition of unity on Ω to write

$$f = \sum_{\nu=1}^{N} f_\nu, \quad f_\nu \in C_c(\Omega_\nu), \quad \mathcal{O}_\nu = \varphi^{-1}(\Omega_\nu), \tag{4}$$

and arrange that each $\mathcal{O}_\nu$ is small enough that there exist $A_\nu \in G\ell(n, \mathbb{R})$ such that

$$\|A_\nu^{-1} D\varphi(x) - I\| \leq \frac{1}{2}, \quad \forall\, x \in \mathcal{O}_\nu. \tag{5}$$

Now set

$$\psi_\nu(x) = A_\nu^{-1} \varphi(x), \tag{6}$$

so we have a diffeomorphism

$$\psi_\nu : \mathcal{O}_\nu \longrightarrow \widetilde{\Omega}_\nu, \quad \widetilde{\Omega}_\nu = A_\nu^{-1}(\Omega_\nu), \tag{7}$$

satisfying

$$\|D\psi_\nu(x) - I\| \le \frac{1}{2}, \quad \forall\, x \in \mathcal{O}_\nu. \tag{8}$$

Now to show that

$$\int_{\mathcal{O}_\nu} f_\nu(\varphi(x)) |\det D\varphi(x)|\, dx = \int_{\Omega_\nu} f_\nu(x)\, dx, \tag{9}$$

it suffices by Lemma A to show that

$$\int_{\Omega_\nu} f_\nu(x)\, dx = \int_{\widetilde{\mathcal{O}}_\nu} f_\nu(\psi_\nu(x)) \det D\psi_\nu(x)\, dx. \tag{10}$$

Let us recast this as a fresh proposition. We simplify the notation by deleting the subscripts ν (and also the tilde from $\widetilde{\mathcal{O}}_\nu$).

Proposition B. *Let $\mathcal{O}$, Ω be open sets in $\mathbb{R}^n$, $\psi : \mathcal{O} \to \Omega$ a C^1 diffeomorphism. Assume*

$$\|D\psi(x) - I\| \le \frac{1}{2}, \quad \forall\, x \in \mathcal{O}. \tag{11}$$

Then, given $f \in C_c(\Omega)$, we have

$$\int_{\mathcal{O}} f(\psi(x)) \det D\psi(x)\, dx = \int_{\Omega} f(x)\, dx. \tag{12}$$

To tackle Proposition B, pick $p \in \mathcal{O}$, set $q = \psi(p)$, and pick $\delta > 0$ such that

$$\overline{B_\delta(p)} \subset \mathcal{O}. \tag{13}$$

As seen in the proof of Corollary 2.2.2A (in §2.2 of this supplement), $\|\psi(x)-\psi(y)\| \ge (1/2)\|x - y\|$, for all $x, y \in \overline{B_\delta(p)}$, so

$$x \in \partial B_\delta(p) \Longrightarrow \|\psi(x) - q\| \ge \frac{\delta}{2}. \tag{14}$$

Hence, by Corollary 2.2.3A (in §2.2 of this supplement),

$$\psi(B_\delta(p)) \supset B_{\delta/2}(q). \tag{15}$$

In fact, $\psi(\overline{B_\delta(p)})$ is compact, so it contains $\overline{B_{\delta/2}(q)}$, and consequently

$$\overline{B_{\delta/2}(q)} \subset \Omega. \tag{16}$$

To continue, we simplify notation by translating axes so that

$$p = 0, \quad q = 0, \tag{17}$$

and we note that (2.3A) (in §2.2 of this supplement) yields

$$x \cdot \psi(x) \geq \frac{1}{2}|x|^2, \quad \text{for} \ \ |x| \leq \delta. \tag{18}$$

Now we pick $\beta \in C_0^\infty(\mathcal{O})$ such that $\beta = 1$ on a neighborhood of $\overline{B_\delta(0)}$, and

$$\operatorname{supp} \beta \subset B_\gamma(0) \subset \mathcal{O}, \tag{19}$$

for some $\gamma > \delta$, and set

$$\Psi(x) = \beta(x)\psi(x) + (1 - \beta(x))x, \quad x \in \mathbb{R}^n. \tag{20}$$

Noting that (18) then holds for $|x| \leq \gamma$, we have

$$x \cdot \Psi(x) = \beta(x) x \cdot \psi(x) + (1 - \beta(x))|x|^2 \geq \frac{1}{2}|x|^2, \quad \forall\, x \in \mathbb{R}^n, \tag{21}$$

hence

$$|\Psi(x)| \geq \frac{1}{2}|x|, \quad \forall\, x \in \mathbb{R}^n, \quad \Psi(x) = x \text{ for } |x| \geq \gamma. \tag{22}$$

In particular,

$$|\Psi(x)| \geq \frac{\delta}{2}, \quad \text{for} \ \ |x| \geq \delta, \tag{23}$$

so

$$\begin{aligned} \operatorname{supp} f \subset B_{\delta/2}(q) &\Rightarrow f \circ \Psi = f \circ \psi, \ \text{supported on } B_\delta(p) \\ &\Rightarrow f(\Psi(x)) \det D\Psi(x) = f(\psi(x)) \det D\psi(x). \end{aligned} \tag{24}$$

Now Theorem 4.5.2 yields

$$\int_{\mathcal{O}} f(\Psi(x)) \det D\Psi(x)\, dx = \int_{\Omega} f(x)\, dx, \tag{25}$$

and we have the following.

Lemma C. *In the setting of Proposition B, if (13)–(16) hold, and if* $\operatorname{supp} f \subset B_{\delta/2}(q)$, *then (12) holds.*

From here, a partition of unity argument finishes the proof of Proposition B.

Alternative proof of Lemma A

Rather than depend on the argument in Proposition 3.1.10, we bring in the following result, of a flavor more like the proof of Theorem 4.5.2, to establish Lemma A.

Lemma D. *Let $A \in M(n,\mathbb{R})$, and assume $f \in C_0^1(\mathbb{R}^n)$. Then*

$$\int_{\mathbb{R}^n} f(e^{tA}x)\,dx = (\det e^{tA})^{-1} \int_{\mathbb{R}^n} f(x)\,dx. \tag{26}$$

Proof. Set

$$v^t(x) = f(e^{tA}x), \quad \text{so} \quad Dv^t(x) = Df(e^{tA}x)e^{tA}. \tag{27}$$

Then

$$\begin{aligned} \frac{d}{dt} v^t(x) &= Df(e^{tA}x)Ae^{tA}x \\ &= Dv^t(x)Ax \\ &= \nabla v^t(x) \cdot Ax. \end{aligned} \tag{28}$$

Hence

$$\varphi(t) = \int f(e^{tA}x)\,dx = \int v^t(x)\,dx \tag{29}$$

satisfies

$$\begin{aligned} \varphi'(t) &= \int \nabla v^t(x) \cdot Ax\,dx \\ &= -\int v^t(x)\,\operatorname{div}(Ax)\,dx, \end{aligned} \tag{30}$$

the last identity by integration by parts. Now, for $A = (a_{jk}) \in M(n,\mathbb{R})$,

$$\operatorname{div}(Ax) = \sum_{j=1}^n \partial_j \sum_{k=1}^n a_{jk}x_k = \operatorname{Tr} A, \tag{31}$$

so

$$\varphi'(t) = -\operatorname{Tr} A\,\varphi(t), \tag{32}$$

hence

$$\varphi(t) = e^{-t \operatorname{Tr} A} \varphi(0), \tag{33}$$

which yields (26), since (cf. §2.3, Exercise 8)

$$e^{-t \operatorname{Tr} A} = \det e^{-tA} = (\det e^{tA})^{-1}. \tag{34}$$

Using Lemma D, we can prove Lemma A as follows. Take $A \in G\ell_+(n, \mathbb{R})$. By Proposition 3.2.11, we can write

$$A = UQ, \quad U \in SO(n),\ Q \in \mathcal{P}(n). \tag{35}$$

Now, by (3.2.19),

$$u = e^{B_1}, \quad \text{for some} \ \ B_1 \in \operatorname{Skew}(n). \tag{36}$$

Meanwhile, Q is conjugate to a diagonal matrix with positive entries, so we also have

$$Q = e^{B_2}, \quad \text{for some} \ \ B_2 \in \operatorname{Sym}(n). \tag{37}$$

Thus

$$\int f(Ax)\, dx = \int f(e^{B_1} e^{B_2} x)\, dx, \tag{38}$$

and the identity (3) follows from two applications of Lemma D.

Finally, if $A \in G\ell(n, \mathbb{R})$ and $\det A < 0$, we can write

$$\begin{aligned} &A = A_1 A_2, \quad A_1 \in G\ell_+(n, \mathbb{R}), \\ &A_2 e_j = -e_j \ \ \text{if} \ \ j = 1,\ e_j \ \text{otherwise}. \end{aligned} \tag{39}$$

Directly verifying Lemma A for A_2 is elementary, and the proof is complete.

Chapter 5. Applications of the Gauss-Green-Stokes formula

§5.1A. Holomorphic inverse function theorem

As defined in §5.1, if $\Omega \subset \mathbb{C}$ is open and $f : \Omega \to \mathbb{C}$, the map f is holomorphic provided f is C^1-smooth and complex differentiable. Equivalently, with

$$f(z) = u(x, y) + iv(x, y), \quad z = x + iy, \quad u, v : \Omega \to \mathbb{R}, \tag{1}$$

the functions u and v are C^1-smooth and satisfy the Cauchy-Riemann equations,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}, \tag{2}$$

derived from

$$\frac{\partial f}{\partial x} = \frac{1}{i}\frac{\partial f}{\partial y}. \tag{3}$$

Here we want to examine the map $F : \Omega \to \mathbb{R}^2$, defined by

$$F(x, y) = \begin{pmatrix} u(x, y) \\ v(x, y) \end{pmatrix}, \tag{4}$$

and compare complex differentiability of f with the behavior of

$$DF(x, y) = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}. \tag{5}$$

Lemma 5.1A.1. *The Cauchy-Riemann equations (2) are equivalent to the statement that*

$$DF(x, y)J = JDF(x, y), \quad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{6}$$

Proof. One routinely computes

$$DF(x, y)J = \begin{pmatrix} u_y & -u_x \\ v_y & -v_x \end{pmatrix}, \quad JDF(x, y) = \begin{pmatrix} -v_x & -v_y \\ u_x & u_y \end{pmatrix}, \tag{7}$$

and checks the equivalence of (2) and (6).

Remark. Given (5), as also see that the CR equations (2) are equivalent to the identity

$$DF(x, y) = \begin{pmatrix} u_x & -v_x \\ v_x & u_x \end{pmatrix} = u_x I + v_x J, \tag{8}$$

and to the identity

$$DF(x,y) = \begin{pmatrix} v_y & u_y \\ -u_y & v_y \end{pmatrix} = v_y I - u_y J. \tag{9}$$

We next look at

$$\det DF(x,y) = u_x v_y - v_x u_y. \tag{10}$$

The CR equations (2) yield

$$\det DF(x,y) = u_x^2 + v_x^2 = v_y^2 + u_y^2. \tag{11}$$

Meanwhile, complex differentiability of f implies

$$\begin{aligned} f'(z) &= f_x = u_x + i v_x, \quad \text{and} \\ f'(z) &= \frac{1}{i} f_y = -i u_y + v_y, \end{aligned} \tag{12}$$

hence

$$|f'(z)|^2 = u_x^2 + v_x^2 = u_y^2 + v_y^2. \tag{13}$$

Comparing (11) and (13), we have:

Lemma 5.1A.2. *If $f : \Omega \to \mathbb{C}$ is holomorphic, given by (1), and if F is given by (4), then*

$$\det DF(x,y) = |f'(z)|^2, \quad z = x + iy \in \Omega. \tag{14}$$

Having these results, we can bring in the inverse function theorem, Theorem 2.2.1, to obtain the following holomorphic inverse function theorem.

Theorem 5.1A.3. *Let $f : \Omega \to \mathbb{C}$ be holomorphic, $z_0 \in \Omega$, $w_0 = f(z_0)$. If $f'(z_0) \neq 0$, then there exist an open neighborhood $\mathcal{O}$ of z_0 in Ω and an open neighborhood U of w_0 in $\mathbb{C}$ such that*

$$f : \mathcal{O} \longrightarrow U \quad \textit{is one-to-one and onto}, \tag{15}$$

with inverse

$$g : U \longrightarrow \mathcal{O}, \tag{16}$$

and g is holomorphic on U.

Proof. By (14), we obtain from Theorem 2.2.1 that such $\mathcal{O}$ and U exist satisfying

$$F : \mathcal{O} \longrightarrow U \quad \text{is one-to-one and onto,} \tag{17}$$

with C^1 inverse

$$G : U \longrightarrow \mathcal{O}. \tag{18}$$

Furthermore, with $(u, v) \in U,\ G(u, v) = (x, y) \in \mathcal{O}$,

$$DG(u, v) = DF(x, y)^{-1}. \tag{19}$$

The commutativity (6) then implies

$$DG(u, v)J = JDG(u, v), \tag{20}$$

so

$$G = \begin{pmatrix} \xi \\ \eta \end{pmatrix}, \quad g = \xi + i\eta \tag{21}$$

gives the holomorphic inverse (16).

Open mapping theorem

We use the holomorphic inverse function theorem to derive the following open mapping theorem for holomorphic functions.

Theorem 5.1A.4. *Let* $\Omega \subset \mathbb{C}$ *be open and connected, and assume* $f : \Omega \to \mathbb{C}$ *is holomorphic and not constant. Then*

$$\mathcal{O} \subset \Omega \text{ open} \Longrightarrow f(\mathcal{O}) \text{ open in } \mathbb{C}. \tag{1}$$

Proof. It suffices to show that each $z_0 \in \Omega$ has a neighborhood $\mathcal{O}_1$ such that

$$f(\mathcal{O}_1) \text{ is a neighborhood of } w_0 = f(z_0). \tag{2}$$

If $f'(z_0) \neq 0$, this follows from the holomorphic inverse function theorem.

Now suppose $f'(z_0) = 0$. Then there is a first $n \in \mathbb{N}$ such that $f^{(n)}(z_0) \neq 0$, so the power series expansion of f about z_0 has the form

$$f(z) = w_0 + \sum_{k=n}^{\infty} a_k (z - z_0)^k, \quad a_n \neq 0, \tag{3}$$

on some disk $D_R(z_0)$. Otherwise, all the power series coefficients would vanish, and f would be constant. We can write

$$f(z) = w_0 + a_n (z - z_0)^n g(z), \tag{4}$$

with

$$g(z) = \sum_{k=0}^{\infty} b_k (z - z_0)^k, \quad b_k = \frac{a_{n+k}}{a_n}, \ b_0 = 1, \tag{5}$$

holomorphic on $D_R(z_0)$. We hence have $g(z)^{1/n}$ holomorphic on some neighborhood $\mathcal{O}_0$ of z_0, and

$$f(z) = w_0 + [b(z - z_0) g(z)^{1/n}]^n = w_0 + H(z)^n, \tag{6}$$

with H holomorphic on $\mathcal{O}_0$,

$$H(z_0) = 0, \quad H'(z_0) \neq 0. \tag{7}$$

Thus, by the holomorphic inverse function theorem, H maps some neighborhood $\mathcal{O}_1$ of z_0 onto a neighborhood U_1 of 0, hence

$$H_n(z) = H(z)^n \text{ maps } \mathcal{O}_1 \text{ onto a neighborhood } U_n \text{ of } 0, \tag{8}$$

and we have (2).

§**5.1B. Exercise on** $\log z$**.**

Exercise 9 of §5.1 introduced $\log : \Omega \to \mathbb{C}$ ($\Omega = \mathbb{C} \setminus (-\infty, 0]$), satisfying

$$\frac{d}{dz} \log z = \frac{1}{z}, \quad \log 1 = 0.$$

This led to the next exercise:

10. Show that

$$e^{\log z} = z, \quad \forall\, z \in \Omega.$$

Two approaches were suggested as hints. Here is a third.
Hint. Set

$$\psi(z) = z e^{-\log z},$$

and show that $\psi'(z) \equiv 0$.

Here is a complementary result.

10A. With Ω as above and $\mathcal{O} = \{z \in \mathbb{C} : |\operatorname{Im} z| < \pi\}$, show that

$$\operatorname{Exp} : \mathcal{O} \longrightarrow \Omega,$$

where $\operatorname{Exp}(z) = e^z$, and that

$$\log e^z = z, \quad \forall\, z \in \mathcal{O}. \tag{*}$$

(*Hint.* Apply d/dz to $\log e^z$ to get (*).) Show that

$$\log : \Omega \longrightarrow \mathcal{O}.$$

§5.1C. The Hodge $*$-operator and harmonic functions

The Hodge $*$-operator is a linear map

$$*:\Lambda^k\mathbb{R}^n \longrightarrow \Lambda^{n-k}\mathbb{R}^n \tag{1C.1}$$

that satisfies the identity

$$u\wedge *v = \langle u,v\rangle \omega, \tag{1C.2}$$

for $u,v\in\Lambda^k\mathbb{R}^n$, where $\langle u,v\rangle$ is the inner product on $\Lambda^k\mathbb{R}^n$ treated in Exercises 5–10 of §4.1 and $\omega = e_1\wedge\cdots\wedge e_n$ ($\{e_j : 1\le j\le n\}$ denoting the standard orthonormal basis of $\mathbb{R}^n$). In particular, we have

$$*1=\omega \tag{1C.3}$$

and

$$*(e_{j_1}\wedge\cdots\wedge e_{j_k}) = (\operatorname{sgn}\pi) e_{\ell_1}\wedge\cdots\wedge e_{\ell_{n-k}}, \tag{1C.4}$$

where $\{j_1,\dots,j_k,\ell_1,\dots,\ell_{n-k}\} = \{1,\dots,n\}$ and π is the permutation taking the first ordered set to the second. We apply this to differential forms, obtaining

$$*:\Lambda^k(\Omega)\longrightarrow\Lambda^{n-k}(\Omega), \tag{1C.5}$$

for open $\Omega\subset\mathbb{R}^n$, upon substituting dx_j for e_j. Thus

$$*1 = dx_1\wedge\cdots\wedge dx_n, \tag{1C.6}$$

and

$$*\sum_j g_j\,dx_j = \sum_j (-1)^{j-1} g_j\,dx_1\wedge\cdots\wedge\widehat{dx_j}\wedge\cdots\wedge dx_n. \tag{1C.7}$$

Here we aim to relate the $*$-operator to harmonic functions and to various formulas that arise in §§4.2–4.4, 5.1, and 5.3.

To begin, let $\Omega\subset\mathbb{R}^n$ be open and let $f:\Omega\to\mathbb{R}$ be smooth of class C^2. Then we have the 1-form

$$df = \sum_j (\partial_j f)\,dx_j, \tag{1C.8}$$

and (1C.7) yields the $(n-1)$-form

$$*df = \sum_j (-1)^{j-1}(\partial_j f)\,dx_1\wedge\cdots\wedge\widehat{dx_j}\wedge\cdots\wedge dx_n. \tag{1C.9}$$

For example,

$$f(x)=\frac{1}{2}|x|^2 \Longrightarrow df=\beta=\sum_{j=1}^{n}(-1)^{j-1}x_j\,dx_1\wedge\cdots\wedge\widehat{dx_j}\wedge\cdots\wedge dx_n, \tag{1C.10}$$

which arose in Exercises 6–9 of §4.3.

To proceed, we have, in general,

$$\begin{aligned} d*df&=\sum_{j,k}(-1)^{j-1}(\partial_k\partial_j f)\,dx_1\wedge\cdots\wedge\widehat{dx_j}\wedge\cdots\wedge dx_n\\ &=\sum_j(\partial_j^2 f)\,dx_1\wedge\cdots\wedge dx_n\\ &=(\Delta f)\omega. \end{aligned} \tag{1C.11}$$

We can interpret this as follows.

Proposition 1C.1. *Given a C^2 smooth $f:\Omega\to\mathbb{R}$, with $\Omega\subset\mathbb{R}^n$ open, the $(n-1)$-form $*df$ is closed if and only if f is harmonic on Ω.*

Let us apply this to the functions $h_n:\mathbb{R}^n\setminus 0\to\mathbb{R}$ given by

$$h_n(x)=\frac{|x|^{2-n}}{2-n},\ n\ge 3,\quad h_2(x)=\log|x|, \tag{1C.12}$$

seen in Exercise 12 of §4.4 and in Exercise 29 of §5.1 to be harmonic on $\mathbb{R}^n\setminus 0$. We have $\partial_j h_n(x)=x_j/|x|^n$, so

$$dh_n=|x|^{-n}\sum_j x_j\,dx_j, \tag{1C.13}$$

hence, with β as in (1C.10),

$$*dh_n=|x|^{-n}\beta, \tag{1C.14}$$

the $(n-1)$-form on $\mathbb{R}^n\setminus 0$ that arises in Exercise 9 of §4.3. This $(n-1)$-form will also arise in Exercise 25 of §5.3. In these two exercises, this form might first seem to be pulled out of a hat, and the fact that is is closed might seem to be a bit of a coincidence. Proposition 1C.1 shows that it arises naturally and why it is closed.

The discussion of harmonic conjugates on planar domains in (5.1.34)–(5.1.35), leading to Proposition 5.1.10, can also be cast in the language of the Hodge star operator. In fact, given a smooth $u:\mathcal{O}\to\mathbb{R},\ \mathcal{O}\subset\mathbb{R}^2$ open,

$$du=\frac{\partial u}{\partial x}\,dx+\frac{\partial u}{\partial y}\,dy\Longrightarrow *du=-\frac{\partial u}{\partial y}\,dx+\frac{\partial u}{\partial x}\,dy=\alpha, \tag{1C.15}$$

the 1-form defined in (5.1.34), so the result that α is closed if and only if u is harmonic is seen to be a special case of Proposition 1A.1.

The Hodge $*$-operator has a natural extension to

$$*:\Lambda^k M\longrightarrow\Lambda^{n-k}M, \tag{1C.16}$$

when M is an oriented n-dimensional manifold with a Riemannian metric. This is introduced in (A.7.63) and plays a significant role in Hodge theory on compact Riemannian manifolds, discussed in Appendix A.7.

§5.2A. Further variations on the change of variable formula for integrals

Here we aim to establish further results on change of variable formulas, using arguments of the flavor of those arising in the proof of Theorem 4.5.2.

Proposition 5.2A.1. *Let $F_t : \mathbb{R}^n \to \mathbb{R}^n$ be a family of smooth diffeomorphisms of $\mathbb{R}^n$, depending smoothly on $t \in I$, an interval in $\mathbb{R}$. Let ω be a smooth, compactly supported n-form on $\mathbb{R}^n$, and set*

$$\varphi(t) = \int\limits_{\mathbb{R}^n} F_t^* \omega. \tag{1}$$

Then $\varphi(t)$ is independent of t.

Proof. As in (5.2.25), define vector fields X_t on $\mathbb{R}^n$ by

$$\frac{d}{dt} F_t(x) = X_t(F_t(x)). \tag{2}$$

Then, by (5.2.26),

$$\frac{d}{dt} F_t^* \omega = F_t^* \mathcal{L}_{X_t} \omega = F_t^* d(\omega \rfloor X_t). \tag{3}$$

Hence

$$\begin{aligned} \varphi'(t) &= \int F_t^* d(\omega \rfloor X_t) \\ &= \int dF_t^* (\omega \rfloor X_t) \\ &= 0, \end{aligned} \tag{4}$$

the last identity by switching to an iterated integral and using the fundamental theorem of calculus.

The following is a stronger result.

Proposition 5.2A.2. *Let $f_t : \mathbb{R}^n \to \mathbb{R}^n$ be a smooth family of maps, depending smoothly on $t \in \mathbb{R}$. Assume that for each compact $K_0 \subset \mathbb{R}^n$, there exist compact $K_1 \subset \mathbb{R}^n$ such that*

$$f_t^{-1}(K_0) \subset K_1, \quad \forall\, t \in \mathbb{R}. \tag{5}$$

Let ω be a smooth, compactly supported n-form on $\mathbb{R}^n$, and set

$$\varphi(t) = \int\limits_{\mathbb{R}^n} f_t^* \omega. \tag{6}$$

Then $\varphi(t)$ is independent of t.

For the proof, we want to replace use of (5.2.26) by arguments arising in the proof of Proposition 5.2.3. To start, take $F : \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}^n,\ F(t,x) = f_t(x)$, and consider

$$\tilde{\omega} = F^*\omega \in \Lambda^n(\mathbb{R}\times\mathbb{R}^n). \tag{7}$$

We have $d\tilde{\omega} = F^* d\omega = 0$. Note that if (5) holds, then

$$\operatorname{supp}\omega \subset K_0 \Longrightarrow \operatorname{supp}\tilde{\omega} \subset \mathbb{R}\times K_1. \tag{8}$$

Now consider

$$\Phi_s : \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}\times\mathbb{R}^n, \quad \Phi_s(t,x) = (s+t,x). \tag{9}$$

We claim that

$$\frac{d}{ds}\Phi_s^*\tilde{\omega} = \Phi_s^* d(\tilde{\omega}\rfloor\partial_t). \tag{10}$$

Indeed, the proof of Lemma 5.2.4 applies to yield this identity. (In the present case, $\ell = n$ and $\mathbb{R}^n$ already provides the desired coordinate system.) Now, with

$$j : \mathbb{R}^n \longrightarrow \mathbb{R}\times\mathbb{R}^n, \quad j(x) = (0,x), \tag{11}$$

we have $F\circ\Phi_s\circ j = f_s$, so

$$\begin{aligned} \frac{d}{ds} f_s^*\omega &= j^* \frac{d}{ds}\Phi_s^*\tilde{\omega} \\ &= j^*\Phi_s^* d(\tilde{\omega}\rfloor\partial_t) \\ &= dj^*\Phi_s^*(\tilde{\omega}\rfloor\partial_t) \\ &= d\alpha_s, \end{aligned} \tag{12}$$

and

$$\operatorname{supp}\omega \subset K_0 \Longrightarrow \operatorname{supp}\alpha_s \subset K_1. \tag{13}$$

Consequently

$$\varphi'(t) = \int_{\mathbb{R}^n} d\alpha_s = 0, \tag{14}$$

again by going to the iterated integral and applying the fundamental theorem of calculus. This proves Proposition 5.2A.2.

§5.3. Further cases of Euler's formula for $\chi(M)$

Let M be a compact 2D manifold. In Exercise 8 of §5.3, we were given a triangulation of M, with

$$V \text{ vertices, } E \text{ edges, and } F \text{ faces,} \tag{1}$$

and obtained the formula

$$\chi(M) = V - E + F \tag{2}$$

by constructing a vector field X with

$$\begin{aligned}&\text{one source in each face, one sink at each vertex,}\\ &\text{and one saddle in each edge.}\end{aligned} \tag{3}$$

Going further, one can suppose M is partitioned into F faces, each of which is a curvilinear polygon, with edges and vertices, and extend formula (2) to this more general setting.

For example, one can take a convex polyhedron $\mathcal{P}$ in $\mathbb{R}^3$, and project its boundary $\partial\mathcal{P}$ onto a sphere $S^2 \subset \mathbb{R}^3$, obtaining the classic Euler formula

$$V - E + F = 2.$$

One can also get variants, such as

$$V - E + F = 0,$$

for a donut shaped polyhedron.

For higher dimensional results, see Chapter 1, §20, especially (20.12), of

M. Taylor, Partial Differential Equations, Vol. 1, Springer, NY, 1996 (2nd ed. 2011).

More exercises

31. Define

$$\varphi : S^2 \longrightarrow \mathbb{R}, \quad \varphi = xyz\big|_{S^2},$$

and consider the vector field $X = \nabla\varphi$ on S^2.

(a) Show that X has 14 critical points, 4 sources, 4 sinks, and 6 saddles.
(b) Show that the count from part (a) is consistent with the calculation $\chi(S^2) = 2$.

Chapter 6. Differential geometry of surfaces

§6.1A. Formulas for P

If $S \subset \mathbb{R}^n$ is a k-dimensional surface, smooth of class C^1, $y \in S$, we have the map $P(y) \in \mathcal{L}(\mathbb{R}^n)$,

$$P(y) = \perp \text{ projection of } \mathbb{R}^n \text{ onto } T_y S, \tag{1}$$

arising in (6.1.16), as an essential tool in the study of differential geometry on S. Here we derive some formulas for $P(y)$, in two settings: either there is a coordinate chart

$$\varphi : \Omega \longrightarrow U \subset S, \quad T_y S = \text{Range } D\varphi(x), \tag{2}$$

with $y = \varphi(x)$, or S is the level set of a map

$$F : \mathcal{O} \longrightarrow \mathbb{R}^\ell \quad (\ell = n - k), \tag{3}$$

where $\mathcal{O}$ is an open neighborhood of S in $\mathbb{R}^n$ and $DF(y) : \mathbb{R}^n \to \mathbb{R}^\ell$ is surjective, so

$$T_y S = \operatorname{Ker} DF(y). \tag{4}$$

In fact, in the former case, we have derived such a formula in §3.5A, but we revisit that work and produce a unified treatment of the two cases.

To get this, it is convenient to bring in the following two linear algebra results.

Lemma 6.1A.1. *Let $A : \mathbb{R}^k \to \mathbb{R}^n$ be injective, and set*

$$P = \perp \textit{ projection of } \mathbb{R}^n \textit{ onto } V = \textit{Range } A. \tag{5}$$

Then

$$P = AG^{-1}A^t, \tag{6}$$

where

$$G = A^t A \in \mathcal{L}(\mathbb{R}^k) \textit{ is positive definite.} \tag{7}$$

Lemma 6.1A.2. *Let $B : \mathbb{R}^n \to \mathbb{R}^\ell$ be surjective, and set*

$$Q = \perp \textit{ projection of } \mathbb{R}^n \textit{ onto } W = \operatorname{Ker} B. \tag{8}$$

Then $I - Q = Q^\perp$ is given by

$$Q^\perp = B^t H^{-1} B, \tag{9}$$

where

$$H = BB^t \in \mathcal{L}(\mathbb{R}^\ell) \textit{ is positive definite.} \tag{10}$$

Proof. The formula (6) follows from a derivation parallel to that leading up to (18) in §3.5A of this supplement. We present a separate verification of (6), as follows. First, for $v \in \mathbb{R}^k$, $Gv \cdot v = \|Av\|^2$, so (7) holds, hence the right side of (6) is well defined; denote it by $\widetilde{P}$. It is routine to check that

$$\widetilde{P}^t = \widetilde{P}, \quad \widetilde{P}^2 = \widetilde{P}, \tag{11}$$

so $\widetilde{P}$ is an orthogonal projection. Furthermore, the hypotheses yield

$$A^t : \mathbb{R}^n \longrightarrow \mathbb{R}^k \text{ is surjective,} \tag{12}$$

hence Range $\widetilde{P} =$ Range A. This yields $\widetilde{P} = P$, and proves Lemma 6.1A.1.

Moving to Lemma 6.1A.2, we see that the hypotheses imply

$$B^t : \mathbb{R}^\ell \longrightarrow \mathbb{R}^n \text{ is injective,} \quad \operatorname{Range} B^t = W^\perp. \tag{13}$$

Thus we can apply Lemma 6.1A.1, with A replaced by B^t (and k replaced by ℓ), hence $G = A^t A$ replaced by $H = BB^t$, and we have from (8) that

$$Q^\perp = \perp \text{ projection of } \mathbb{R}^n \text{ onto } W^\perp. \tag{14}$$

Hence the formula (6) yields the formula (9) for $Q^\perp$, and Lemma 6.1A.2 is proved.

We can now present our formulas for $P(y)$.

Proposition 6.1A.3. *Assume there is a coordinate chart on S, as in (2), with $y = \varphi(x) \in U$. Then*

$$P(y) = D\varphi(x)\, G(x)^{-1}\, D\varphi(x)^t, \tag{15}$$

with

$$G(x) = D\varphi(x)^t D\varphi(x). \tag{16}$$

Proposition 6.1A.4. *Assume S is a level set of F, as in (3), with $DF(y) : \mathbb{R}^n \to \mathbb{R}^\ell$ surjective. Then*

$$I - P(y) = DF(y)^t \, H(y)^{-1} \, DF(y), \tag{17}$$

with

$$H(y) = DF(y) DF(y)^t. \tag{18}$$

These results are direct consequences of Lemma 6.1A.1 and 6.1A.2, with $A = D\varphi(x)$ and $B = DF(y)$, respectively.

Remark. A standard presentation of Proposition 6.1A.4 in case $\ell = 1$ is to set $N(y) = \nabla F(y)/|\nabla F(y)|$ (unit normal to S at y) and write

$$P(y)^\perp v = (N(y) \cdot v) N(y), \tag{19}$$

or equivalently

$$P(y)^\perp = N(y) N(y)^t. \tag{20}$$

We leave it to the reader to show how (17) leads to this formula in such a case.

§6.2A. Curvature and torsion of curves in $\mathbb{R}^3$.

Given a curve $c(t) = (x(t), y(t), z(t))$ in 3-space, we define its velocity and acceleration by

$$v(t) = c'(t), \quad a(t) = v'(t) = c''(t). \tag{2A.1}$$

We also define its speed $s'(t)$ and arclength by

$$s'(t) = \|v(t)\|, \quad s(t) = \int_{t_0}^{t} s'(\tau)\, d\tau, \tag{2A.2}$$

assuming we start at $t = t_0$. We define the unit tangent vector to the curve as

$$T(t) = \frac{v(t)}{\|v(t)\|}. \tag{2A.3}$$

Henceforth we assume the curve is parametrized by arclength.

We define the *curvature* $\kappa(s)$ of the curve and the normal $N(s)$ by

$$\kappa(s) = \left\|\frac{dT}{ds}\right\|, \quad \frac{dT}{ds} = \kappa(s)N(s). \tag{2A.4}$$

Note that

$$T(s) \cdot T(s) = 1 \Longrightarrow T'(s) \cdot T(s) = 0, \tag{2A.5}$$

so indeed $N(s)$ is orthogonal to $T(s)$. We then define the binormal $B(s)$ by

$$B(s) = T(s) \times N(s). \tag{2A.6}$$

For each s, the vectors $T(s)$, $N(s)$ and $B(s)$ are mutually orthogonal unit vectors, known as the Frenet frame for the curve $c(s)$. Rules governing the cross product yield

$$T(s) = N(s) \times B(s), \quad N(s) = B(s) \times T(s). \tag{2A.7}$$

(For material on the cross product, see the exercises at the end of §1.4.)

The *torsion* of a curve measures the change in the plane generated by $T(s)$ and $N(s)$, or equivalently it measures the rate of change of $B(s)$. Note that, parallel to (2A.5),

$$B(s) \cdot B(s) = 1 \Longrightarrow B'(s) \cdot B(s) = 0.$$

Also, differentiating (2A.6) and using (2A.4), we have

$$B'(s) = T'(s) \times N(s) + T(s) \times N'(s) = T(s) \times N'(s) \Longrightarrow B'(s) \cdot T(s) = 0. \tag{2A.8}$$

We deduce that $B'(s)$ is parallel to $N(s)$. We define the torsion by

$$\frac{dB}{ds} = -\tau(s)N(s). \tag{2A.9}$$

We complement the formulas (2A.4) and (2A.9) for dT/ds and dB/ds with one for dN/ds. Since $N(s) = B(s) \times T(s)$, we have

$$\frac{dN}{ds} = \frac{dB}{ds} \times T + B \times \frac{dT}{ds} = \tau N \times T + \kappa B \times N, \tag{2A.10}$$

or

$$\frac{dN}{ds} = -\kappa(s)T(s) + \tau(s)B(s). \tag{2A.11}$$

Together, (2A.4), (2A.9) and (2A.11) are known as the Frenet-Serret formulas.

Example. Pick $a, b > 0$ and consider the helix

$$c(t) = (a\cos t, a\sin t, bt). \tag{2A.12}$$

Then $v(t) = (-a\sin t, a\cos t, b)$ and $\|v(t)\| = \sqrt{a^2+b^2}$, so we can pick $s = t\sqrt{a^2+b^2}$ to parametrize by arc length. We have

$$T(s) = \frac{1}{\sqrt{a^2+b^2}}(-a\sin t, a\cos t, b), \tag{2A.13}$$

hence

$$\frac{dT}{ds} = \frac{1}{a^2+b^2}(-a\cos t, -a\sin t, 0). \tag{2A.14}$$

By (2A.4), this gives

$$\kappa(s) = \frac{a}{a^2+b^2}, \quad N(s) = (-\cos t, -\sin t, 0). \tag{2A.15}$$

Hence

$$B(s) = T(s) \times N(s) = \frac{1}{\sqrt{a^2+b^2}}(b\sin t, -b\cos t, a). \tag{2A.16}$$

Then

$$\frac{dB}{ds} = \frac{1}{a^2+b^2}(b\cos t, b\sin t, 0), \tag{2A.17}$$

so, by (2A.9),

$$\tau(s) = \frac{b}{a^2 + b^2}. \tag{2A.18}$$

In particular, for the helix (2A.12), we see that the curvature and torsion are *constant.*

Let us collect the Frenet-Serret equations

$$\begin{aligned} \frac{dT}{ds} &= \qquad \kappa N \\ \frac{dN}{ds} &= -\kappa T \qquad\qquad + \tau B \\ \frac{dB}{ds} &= \qquad - \tau N \end{aligned} \tag{2A.19}$$

for a smooth curve $c(s)$ in $\mathbb{R}^3$, parametrized by arclength, with unit tangent $T(s)$, normal $N(s)$, and binormal $B(s)$, given by

$$N(s) = \frac{1}{\kappa(s)} T'(s), \quad B(s) = T(s) \times N(s), \tag{2A.20}$$

assuming $\kappa(s) = \|T'(s)\| > 0$.

The basic existence and uniqueness theory, given in §2.3, applies to (2A.19). If $\kappa(s)$ and $\tau(s)$ are given smooth functions on an interval $I = (a, b)$ and $s_0 \in I$, then, given $T_0, N_0, B_0 \in \mathbb{R}^3$, (2A.19) has a unique solution on $s \in I$ satisfying

$$T(s_0) = T_0, \quad N(s_0) = N_0, \quad B(s_0) = B_0. \tag{2A.21}$$

We now establish the following.

Proposition 2A.1. *Assume κ and τ are given smooth functions on I, with $\kappa > 0$ on I. Assume $\{T_0, N_0, B_0\}$ is an orthonormal basis of $\mathbb{R}^3$, such that $B_0 = T_0 \times N_0$. Then there exists a smooth, unit-speed curve $c(s)$, $s \in I$, for which the solution to (2A.19) and (2A.21) is the Frenet frame.*

To construct the curve, take $T(s), N(s)$, and $B(s)$ to solve (2A.19) and (2A.21), pick $p \in \mathbb{R}^3$ and set

$$c(s) = p + \int_{s_0}^{s} T(\sigma)\, d\sigma, \tag{2A.22}$$

so $T(s) = c'(s)$ is the velocity of this curve. To deduce that $\{T(s), N(s), B(s)\}$ is the Frenet frame for $c(s)$, for all $s \in I$, we need to know:

$$\{T(s), N(s), B(s)\} \ \text{orthonormal, with} \ B(s) = T(s) \times N(s), \quad \forall\, s \in I. \tag{2A.23}$$

In order to pursue the analysis further, it is convenient to form the 3×3 matrix-valued function

$$F(s) = (T(s), N(s), B(s)), \tag{2A.24}$$

whose *columns* consist respectively of $T(s)$, $N(s)$, and $B(s)$. Then (2A.23) is equivalent to

$$F(s) \in SO(3), \quad \forall\, s \in I, \tag{2A.25}$$

with $SO(3)$ defined as in §1.4. The hypothesis on $\{T_0, N_0, B_0\}$ stated in Proposition 7.1 is equivalent to $F_0 = (T_0, N_0, B_0) \in SO(3)$. Now $F(s)$ satisfies the differential equation

$$F'(s) = F(s)A(s), \quad F(s_0) = F_0, \tag{2A.26}$$

where

$$A(s) = \begin{pmatrix} 0 & -\kappa(s) & 0 \\ \kappa(s) & 0 & -\tau(s) \\ 0 & \tau(s) & 0 \end{pmatrix}. \tag{2A.27}$$

Note that

$$\frac{dF^*}{ds} = A(s)^*F(s)^* = -A(s)F(s)^*, \tag{2A.28}$$

since $A(s)$ in (2A.27) is skew-adjoint. Hence

$$\begin{aligned} \frac{d}{ds}\, F(s)F(s)^* &= \frac{dF}{ds}F(s)^* + F(s)\frac{dF^*}{ds} \\ &= F(s)A(s)F(s)^* - F(s)A(s)F(s)^* \\ &= 0. \end{aligned} \tag{2A.29}$$

Thus, whenever (2A.26)–(2A.27) hold,

$$F_0F_0^* = I \Longrightarrow F(s)F(s)^* \equiv I, \tag{2A.30}$$

and we have (2A.23).

Let us specialize the system (2A.19), or equivalently (2A.26), to the case where κ and τ are *constant*, i.e.,

$$F'(s) = F(s)A, \quad A = \begin{pmatrix} 0 & -\kappa & 0 \\ \kappa & 0 & -\tau \\ 0 & \tau & 0 \end{pmatrix}, \tag{2A.31}$$

with solution

$$F(s) = F_0\, e^{(s-s_0)A}. \tag{2A.32}$$

We have already seen in that a helix of the form (2A.12) has curvature κ and torsion τ, with

$$\kappa = \frac{a}{a^2+b^2}, \quad \tau = \frac{b}{a^2+b^2}, \tag{2A.33}$$

and hence

$$a = \frac{\kappa}{\kappa^2+\tau^2}, \quad b = \frac{\tau}{\kappa^2+\tau^2}. \tag{2A.34}$$

In (2A.12), s and t are related by $t = s\sqrt{\kappa^2+\tau^2}$.

We can also see such a helix arise via a direct calculation of e^{sA}, which we now produce. First, a straightforward calculation gives, for A as in (2A.31),

$$\det(\lambda I - A) = \lambda(\lambda^2+\kappa^2+\tau^2), \tag{2A.35}$$

hence

$$\operatorname{Spec}(A) = \{0, \pm i\sqrt{\kappa^2+\tau^2}\}. \tag{2A.36}$$

An inspection shows that we can take

$$v_1 = \frac{1}{\sqrt{\kappa^2+\tau^2}}\begin{pmatrix}\tau\\0\\\kappa\end{pmatrix}, \quad v_2 = \begin{pmatrix}0\\1\\0\end{pmatrix}, \quad v_3 = \frac{1}{\sqrt{\kappa^2+\tau^2}}\begin{pmatrix}-\kappa\\0\\\tau\end{pmatrix}, \tag{2A.37}$$

and then

$$Av_1 = 0, \quad Av_2 = \sqrt{\kappa^2+\tau^2}\, v_3, \quad Av_3 = -\sqrt{\kappa^2+\tau^2}\, v_2. \tag{2A.38}$$

In particular, with respect to the basis $\{v_2, v_3\}$ of $V = \operatorname{Span}\{v_2, v_3\}$, $A|_V$ has the matrix representation

$$B = \sqrt{\kappa^2+\tau^2}\begin{pmatrix}0 & -1\\1 & 0\end{pmatrix}. \tag{2A.39}$$

We see that

$$e^{sA}v_1 = v_1, \tag{2A.40}$$

while

$$
\begin{aligned}
e^{sA}v_2 &= (\cos s\sqrt{\kappa^2+\tau^2})v_2 \quad +(\sin s\sqrt{\kappa^2+\tau^2})v_3, \\
e^{sA}v_3 &= -(\sin s\sqrt{\kappa^2+\tau^2})v_2+(\cos s\sqrt{\kappa^2+\tau^2})v_3.
\end{aligned} \tag{2A.41}
$$

Exercises

1. Consider a curve $c(t)$ in $\mathbb{R}^3$, not necessarily parametrized by arclength. Show that the acceleration $a(t)$ is given by

$$
a(t) = \frac{d^2s}{dt^2}T + \kappa\Bigl(\frac{ds}{dt}\Bigr)^2 N. \tag{2A.42}
$$

Hint. Differentiate $v(t) = (ds/dt)T(t)$ and use the chain rule $dT/dt = (ds/dt)(dT/ds)$, plus (2A.4).

2. Show that

$$
\kappa B = \frac{v \times a}{\|v\|^3}. \tag{2A.43}
$$

Hint. Take the cross product of both sides of (2A.42) with T, and use (2A.6).

3. In the setting of Exercises 1–2, show that

$$
\kappa^2\tau\|v\|^6 = -a\cdot(v\times a'). \tag{2A.44}
$$

Deduce from (2A.43)–(2A.44) that

$$
\tau = \frac{(v\times a)\cdot a'}{\|v\times a\|^2}. \tag{2A.45}
$$

Hint. Proceed from (2A.43) to

$$
\frac{d}{dt}(\kappa\|v\|^3)\,B + \kappa\|v\|^3\frac{dB}{dt} = \frac{d}{dt}(v\times a) = v\times a',
$$

and use $dB/dt = -\tau(ds/dt)N$, as a consequence of (2A.9). Then dot with a, and use $a\cdot N = \kappa\|v\|^2$, from (2A.42), to get (2A.44).

4. Consider the curve $c(t)$ in $\mathbb{R}^3$ given by

$$
c(t) = (a\,\cos t, b\,\sin t, t),
$$

where a and b are given positive constants. Compute the curvature, torsion, and Frenet frame.
Hint. Use (2A.43) to compute κ and B. Then use $N = B \times T$. Use (2A.45) to compute τ.

5. Suppose c and $\tilde{c}$ are two curves, both parametrized by arc length over $0 \leq s \leq L$, and both having the same curvature $\kappa(s) > 0$ and the same torsion $\tau(s)$. Show that there exit $x_0 \in \mathbb{R}^3$ and $A \in O(3)$ such that

$$\tilde{c}(s) = Ac(s) + x_0, \quad \forall\ s \in [0, L].$$

Hint. To begin, show that if their Frenet frames coincide at $s = 0$, i.e., $\widetilde{T}(0) = T(0)$, $\widetilde{N}(0) = N(0)$, $\widetilde{B}(0) = B(0)$, then $\widetilde{T} \equiv T$, $\widetilde{N} \equiv N$, $\widetilde{B} \equiv B$.

6. Suppose c is a curve in $\mathbb{R}^3$ with curvature $\kappa > 0$. Show that there exists a plane in which $c(t)$ lies for all t if and only if $\tau \equiv 0$.
Hint. When $\tau \equiv 0$, the plane should be parallel to the orthogonal complement of B.

§6.4. New endgame to proof of Proposition 6.4.19

Let G be a smooth matrix group with a bi-invariant metric tensor. As seen in Proposition 6.4.18, for each $g \in G$,

$$\psi_g : G \longrightarrow G, \quad \psi_g(x) = gx^{-1}g, \tag{6.4.109}$$

is an isometry of G, fixing g and satisfying

$$D\psi_g(g) = -I \ \text{ on } \ T_gG. \tag{6.4.110}$$

Here we aim to provide a new endgame to the proof of the following.

Proposition 6.4.19. *If γ is a unit speed geodesic on G satisfying $\gamma(0) = I$, then*

$$\gamma(s+t) = \gamma(s)\gamma(t). \tag{6.4.111}$$

Proof. Fix $t \in \mathbb{R}$ and consider

$$\sigma(s) = \gamma(t+s). \tag{6.4.112}$$

This is a unit-speed geodesic satisfying $\sigma(0) = \gamma(t),\ \sigma'(0) = \gamma'(t)$. It follows from Proposition 6.4.18 that

$$\tilde{\sigma}(s) = \psi_{\gamma(t)}(\sigma(s)) \tag{6.4.113}$$

is the unit-speed geodesic satisfying $\tilde{\sigma}(0) = \gamma(t),\ \tilde{\sigma}'(0) = -\gamma'(t)$. This forces $\tilde{\sigma}(s) = \gamma(t-s)$, i.e.,

$$\begin{aligned} \gamma(t-s) &= \psi_{\gamma(t)}(\gamma(t+s)) \\ &= \gamma(t)\gamma(t+s)^{-1}\gamma(t). \end{aligned} \tag{6.4.114}$$

Taking $t = 0$ gives

$$\gamma(-s) = \gamma(s)^{-1}, \tag{6.4.115}$$

and then taking $s \mapsto -s$ gives

$$\gamma(s+t) = \gamma(t)\gamma(s-t)\gamma(t). \tag{6.4.116}$$

Taking $s = t$ gives $\gamma(2t) = \gamma(t)^2$, and then we obtain by induction that

$$\gamma((n+1)t) = \gamma(t)\gamma((n-1)t)\gamma(t) = \gamma(t)^{n+1}, \tag{6.4.117}$$

for each $n \in \mathbb{N}$. A limiting argument gives (6.4.111) when s and t have the same sign. In such a case, (6.4.114) gives

$$\gamma(t-s) = \gamma(t)\gamma(s)^{-1}\gamma(t)^{-1}\gamma(t) = \gamma(t)\gamma(-s), \tag{6.4.117A}$$

so we have (6.4.111) in general.

Chapter 7. Fourier analysis

§7.4. Comment on the dimensions of spaces of spherical harmonics

Section 7.4 deals with the space V_k of spherical harmonics of degree k on S^{n-1}, seen to be isomorphic to the space

$$\begin{aligned}\mathcal{H}_k = &\text{ space of harmonic polynomials on } \mathbb{R}^n, \\ &\text{ homogeneous of degree } k.\end{aligned} \tag{1}$$

It is shown in (7.4.49) that

$$\dim \mathcal{H}_k = \dim \mathcal{P}^k(\mathbb{R}^{n-2}) + \dim \mathcal{P}^{k-1}(\mathbb{R}^{n-2}), \tag{2}$$

where

$$\mathcal{P}^k(\mathbb{R}^n) = \text{ space of polynomials on } \mathbb{R}^n, \text{ of degree} \le k. \tag{3}$$

The formula

$$\dim \mathcal{P}^k(\mathbb{R}^n) = \binom{n+k}{k} \tag{4}$$

leads to the computation that, on S^{n-1},

$$\dim V_k = \binom{k+n-2}{k} + \binom{k+n-3}{k-1}. \tag{5}$$

See (7.4.52).

Here we point out that a nice derivation of (4) is given in the supplement "Further power series exercises," for §2.1. See Exercise 5 there.

An alternative formula for $\dim V_k$ appears in (7.4.123).

Appendix A. Complementary material

§A.7. Correction of formula (A.7.39)

Formula (A.7.39) should be revised to read as follows:

$$-(\Delta u, u)_{L^2} = \|du\|_{L^2}^2 + \|\delta u\|_{L^2}^2, \quad u \in \Lambda^k M. \tag{A.7.39}$$

That is, replace $\|\Delta u\|_{L^2}^2$ on the left side of the original (A.7.32) by $(-\Delta u, u)_{L^2}$. This is what actually follows from (A.7.36). With this change, none of the surrounding arguments are affected.

§A.7. Topological invariance of de Rham cohomology

If X is a smooth compact manifold, the definition of the de Rham cohomology groups $\mathcal{H}^k(X)$ depends explicitly on the differential structure of X. In light of this, it is of interest that the following topological invariance result holds.

Proposition T.1. *Let X and Y be smooth, compact, n-dimensional manifolds, and let*

$$f : X \longrightarrow Y \quad \text{be a homeomorphism.} \tag{T.1}$$

Then f induces an isomorphism of de Rham cohomology,

$$f^* : \mathcal{H}^k(Y) \longrightarrow \mathcal{H}^k(X), \quad \text{for } 0 \le k \le n. \tag{T.2}$$

Part of the significance of this result lies in the fact that there are compact smooth manifolds that are homeomorphic but not diffeomorphic. Indeed, [Mil] stunned the mathematical world by producing smooth manifolds homeomorphic but not diffeomorphic to S^7.

Proof of Proposition T.1. First, embedding Y smoothly in some Euclidean space, we can find a sequence of C^∞ maps $\varphi_\nu : X \to Y$ such that $\varphi_\nu \to f$ uniformly as $\nu \to \infty$. Similarly, with $g = f^{-1} : Y \to X$, we can find C^∞ maps $\psi_\nu : Y \to X$ such that $\psi_\nu \to g$ uniformly. It follows that

$$\psi_\nu \circ \varphi_\nu : X \to X \quad \text{and} \quad \varphi_\nu \circ \psi_\nu : Y \to Y \tag{T.3}$$

are smooth maps and $\psi_\nu \circ \varphi_\nu$ and $\varphi_\nu \circ \psi_\nu$ uniformly tend to the identity maps on X and Y, respectively. Of course, we have the induced maps

$$\varphi_\nu^* : \mathcal{H}^k(Y) \to \mathcal{H}^k(X), \quad \psi_\nu^* : \mathcal{H}^k(X) \to \mathcal{H}^k(Y), \tag{T.4}$$

for $0 \le k \le n$, hence

$$\begin{aligned} \varphi_\nu^* \circ \psi_\nu^* &= (\psi_\nu \circ \varphi_\nu)^* : \mathcal{H}^k(X) \longrightarrow \mathcal{H}^k(X), \\ \psi_\nu^* \circ \varphi_\nu^* &= (\varphi_\nu \circ \psi_\nu)^* : \mathcal{H}^k(Y) \longrightarrow \mathcal{H}^k(Y). \end{aligned} \tag{T.5}$$

The key to the endgame is very simple. There exists N such that for $\nu \ge N$, $\psi_\nu \circ \varphi_\nu$ and $\varphi_\nu \circ \psi_\nu$ in (T.3) are smoothly homotopic to the identity maps on X and Y, respectively, so the induced maps on cohomology are the identity maps. That is, the maps in (T.5) are the identity maps, for $\nu \ge N$. Hence, for $\nu \ge N$,

$$\varphi_\nu^* : \mathcal{H}^k(Y) \xrightarrow{\approx} \mathcal{H}^k(X), \quad \psi_\nu^* : \mathcal{H}^k(X) \xrightarrow{\approx} \mathcal{H}^k(Y), \tag{T.6}$$

these maps being 2-sided inverses of each other.

We also note that, for N large enough,

$$\begin{aligned} \mu,\nu \geq N &\Rightarrow \varphi_\nu, \varphi_\mu \text{ smoothly homotopic, ditto for } \psi_\nu, \psi_\mu \\ &\Rightarrow \varphi_\nu^* = \varphi_\mu^* \text{ and } \psi_\nu^* = \psi_\mu^* \text{ in (T.4)}, \end{aligned} \tag{T.7}$$

so the isomorphisms (T.6) are uniquely determined by the map f.

There are singular cohomology groups $\mathcal{H}^k_{\rm sing}(X,\mathbb{R})$, designed to be topological invariants, and a result known as de Rham's theorem yields a natural isomorphism

$$\mathcal{H}^k(X) \approx \mathcal{H}^k_{\rm sing}(X,\mathbb{R}), \tag{T.8}$$

when X is a smooth compact manifold. A proof can be found in [Lee].

References

[Lee] J. Lee, Introduction to Smooth Manifolds, Springer NY, 2013.
[Mil] J. Milnor, On manifolds homeomorphic to the 7-sphere, Annals of Math. 64 (1956), 399-405.