$\mathcal{D}(M)/\mathcal{D}_X(M)$

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Let M be a smooth, compact, n-dimensional manifold. It will be convenient to endow M with a Riemannian metric. Assume that done. Let $\mathcal{D}(M)$ denote the group of C^∞ diffeomorphisms of M. Let $X \subset M$, and denote by $\mathcal{D}_X(M)$ the set of $F \in \mathcal{D}(M)$ such that $F(X) = X$.

Claim 1. If X is a smooth, compact k-dimensional submanifold of M, then $\mathcal{D}(M)/\mathcal{D}_X(M)$ has the structure of a smooth Frechet manifold, with coordinate charts based on the Frechet space

$$C^\infty(X, N^*X), \tag{1}$$

of smooth sections of the conormal bundle N^*X.

Let's first say something about $\mathcal{D}(M)$ as a smooth Frechet Lie group, with coordinate charts based on the Frechet space $\mathrm{Vect}(M)$ of smooth vector fields on M. One tempting candidate for a coordinate chart about the identity element $I \in \mathcal{D}(M)$ (which, however, does not work) is

$$\mathcal{EXP} : \mathrm{Vect}(M) \longrightarrow \mathcal{D}(M), \quad \mathcal{EXP}(V)(x) = \Phi_V^1(x), \tag{2}$$

where Φ_V^t is the flow on M generated by V, i.e., $x_V(t) = \Phi_V^t(x)$ satisfies

$$x_V'(t) = V(x_V(t)), \quad x_V(0) = x. \tag{3}$$

It turns out that $\mathcal{EXP}$ does not map a neighborhood of 0 in $\mathrm{Vect}(M)$ onto a neighborhood of I in $\mathcal{D}(M)$. A better candidate is

$$E : \mathcal{O} \longrightarrow \mathcal{D}(M), \quad E(V)(x) = \mathrm{Exp}_x\, V(x), \tag{4}$$

where $\mathcal{O}$ is a sufficiently small neighborhood of 0 in $\mathrm{Vect}(M)$ and

$$\mathrm{Exp}_x : T_xM \longrightarrow M \tag{5}$$

arises from the geodesic flow on TM (bringing in that metric tensor).

On to $\mathcal{D}(M)/\mathcal{D}_X(M)$. Let us form $S_X(M)$, the set of all smooth submanifolds $Y \subset M$ that are diffeomorphic to X, and the subset

$$S_X^\#(M) \subset S_X(M), \tag{6}$$

consisting of Y such that $Y = F(X)$ for some $F \in \mathcal{D}(M)$. The group $\mathcal{D}(M)$ acts transitively on $S_X^\#(M)$, and the subgroup fixing X is $\mathcal{D}_X(M)$, so we have a natural bijection

$$\mathcal{D}(M)/\mathcal{D}_X(M) \longrightarrow S_X^\#(M). \tag{7}$$

We aim to give $S_X^\#(M)$ the structure of a smooth Frechet manifold. Actually, we will give $S_X(M)$ the structure of a smooth Frechet manifold, and bring in the following.

Claim 2. $S_X^\#(M)$ is an open subset of $S_X(M)$.
Proof. Later, maybe. (If this falls through, it's back to the drawing board.)

At this point, we use the metric tensor on M to produce an isomorphism of N^*X with the normal bundle NX. We define a map

$$\Psi : \mathcal{O} \longrightarrow S_X(M), \quad \mathcal{O} \subset C^\infty(X, NX), \tag{8}$$

a neighborhood of 0, as follows. Given $u \in C^\infty(X, NX)$, consider

$$\psi(u) : X \longrightarrow M, \quad \psi(u)(x) = \operatorname{Exp}_x u(x), \tag{9}$$

with Exp_x as in (5). There exists a neighborhood $\mathcal{O}$ of 0 in $C^\infty(X, NX)$ such that $\psi(u)$ maps X diffeomorphically onto its image for each $u \in \mathcal{O}$, and we define

$$\Psi(u) = \psi(u)(X) = \{\psi(u)(x) : x \in X\} \subset M. \tag{10}$$

Claim 3. There exists a neighborhood $\mathcal{O}$ of 0 in $C^\infty(X, NX)$ such that $\Psi : \mathcal{O} \to S_X(M)$ provides a smooth coordinate chart of $\mathcal{O}$ onto a neighborhood of X in $S_X(M)$.

Reference

The following is perhaps the best source on infinite-dimensional Frechet manifolds.

R. Hamilton, The inverse function theorem of Nash and Moser, Bull. AMS 7 (1982), 65–222.