Modular Forms and Automorphic Forms

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1. Introduction

To begin, consider the upper half-plane $H = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$. The group $S\ell(2,\mathbb{R})$ acts on H by linear fractional transformations:

$$T(z) = \frac{az+b}{cz+d}, \quad T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad ad - bc = 1. \tag{1.1}$$

Let $\Gamma \subset S\ell(2,\mathbb{R})$ be a discrete subgroup. The paradigmatic example is $S\ell(2,\mathbb{Z})$. More generally, one is particularly interested in groups Γ of finite index in $S\ell(2,\mathbb{Z})$. A modular form of weight 0 for Γ is a holomorphic function f on H that is invariant under Γ,

$$f(Tz) = f(z), \quad \forall\, T \in \Gamma,\ z \in H, \tag{1.2}$$

so it is well defined on $\Gamma \setminus H$. We assume f has the additional property that $|f(z)|$ is polynomially bounded in y. The notion of a modular form of nonzero weight is often introduced by a formula that might leave the uninitiated cold, so we approach it gently.

One example of the sort of form we are interested in is the class of holomorphic differential forms

$$f(z)\, dz. \tag{1.3}$$

The invariance we seek, parallel to (1.2), is

$$T^*(f\, dz) = f\, dz, \quad \forall\, T \in \Gamma. \tag{1.4}$$

Now

$$T^* dz = T'(z)\, dz, \tag{1.5}$$

and

$$T'(z) = \frac{a(cz+d) - (az+b)c}{(cz+d)^2} = \frac{ad-bc}{(cz+d)^2} = \frac{1}{(cz+d)^2}, \tag{1.6}$$

given T as in (1.1). Hence the invariance (1.4) is equivalent to

$$f\Bigl(\frac{az+b}{cz+d}\Bigr) = (cz+d)^2 f(z), \quad \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma. \tag{1.7}$$

If f is holomorphic on H and satisfies (1.7), and if $y|f(z)|$ is polynomially bounded in y, we say $f\,dz$ (or, briefly, f) is a modular form of weight 2.

More generally, given $k \in \mathbb{Z}$ and f holomorphic on H, we consider

$$f(z)\,(dz)^k, \tag{1.8}$$

whose invariance under Γ becomes

$$f(Tz)(T^*dz)^k = f(z)\,(dz)^k, \tag{1.9}$$

or

$$f(z) = T'(z)^k f(Tz), \quad \forall\, T \in \Gamma, \tag{1.10}$$

or equivalently

$$f\Bigl(\frac{az+b}{cz+d}\Bigr) = (cz+d)^{2k} f(z), \quad \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma. \tag{1.11}$$

If f is holomorphic on H and satisfies (1.11), and if $y^k|f(z)|$ is polynomially bounded in y, we say $f\,(dz)^k$ (or, briefly, f) is a modular form of weight $2k$. Note that $y^k|f(z)|$ is invariant under the action of Γ if f satisfies (1.11), since

$$\operatorname{Im}(Tz) = (\operatorname{Im} z)|cz+d|^{-2}. \tag{1.12}$$

For general $k \in \mathbb{Z}$, we might try to say a holomorphic function f on H defines a modular function of weight k if

$$f\Bigl(\frac{az+b}{cz+d}\Bigr) = (cz+d)^{k} f(z), \quad \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma, \tag{1.13}$$

and $y^{k/2}|f(z)|$ is polynomially bounded in y. However, if $A \in S\ell(2,\mathbb{Z})$, so is $-A$. The left side of (1.13) is unchanged if we replace A by $-A$, but the right side changes sign if k is odd. A more subtle characterization of forms of odd weight is called for.

To approach this, note that the objects defined by (1.3) and (1.8) are sections of homogeneous complex line bundles over H. This presents modular forms of even weight as sections of such line bundles. We can extend this presentation to modular forms of odd weight, as follows.

Let us start with a geometrical description of the complex line bundles for which the forms (1.8) are sections, line bundles naturally associated to the unit frame bundle of H, endowed with its Poincaré metric,

$$ds_H = \frac{1}{y}|dz|. \tag{1.14}$$

The group $S\ell(2,\mathbb{R})$ acts as a group of isometries on H, via (1.1), and we have

$$H \approx S\ell(2,\mathbb{R})/K, \tag{1.15}$$

where

$$K = \{T \in S\ell(2,\mathbb{R}) : T(i) = i\} = \Big\{R(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} : \theta \in \mathbb{R}/2\pi\mathbb{Z}\Big\}. \tag{1.16}$$

Now I and $-I$ both act trivially on H, so we also have

$$H \approx PS\ell(2,\mathbb{R})/K_0, \quad PS\ell(2,\mathbb{R}) = S\ell(2,\mathbb{R})/\{\pm I\}, \quad K_0 = K/\{\pm I\}, \tag{1.17}$$

and $PS\ell(2,\mathbb{R})$ is naturally equivalent to the unit frame bundle of H. This is a principal K_0-bundle over H, and the irreducible representations of K_0 give rise to the various line bundles over H associated to (1.8).

Meanwhile, $S\ell(2,\mathbb{R})$ is a principal K-bundle over H, lifting the bundle $PS\ell(2,\mathbb{R})$ and the group K_0. (This endows H with a natural spin structure.) The irreducible representations of K give rise to a larger family of complex line bundles over H. Some of these representations arise from those of K_0, via the projection

$$K \longrightarrow K_0. \tag{1.18}$$

To be specific, the irreducible representations of K are given by

$$\alpha_k(R(\theta)) = e^{-ik\theta}, \quad k \in \mathbb{Z}, \tag{1.19}$$

and those that arise from K_0 are those for which k is even. The forms (1.8) are sections of the line bundle associated with the representation α_{2k}.

As is standard, there is a natural isomorphism between the space of sections of the line bundle L_k on H associated to the representation α_k of K and the space of functions $g : S\ell(2,\mathbb{R}) \to \mathbb{C}$ satisfying

$$g(SR(\theta)) = \alpha_k(R(\theta))^{-1} g(S), \quad \forall\, S \in S\ell(2,\mathbb{R}),\ \theta \in \mathbb{R}/2\pi\mathbb{Z}. \tag{1.20}$$

Consequently there is a natural left action of $S\ell(2,\mathbb{R})$ (hence of Γ) on the sections of these line bundles, induced by

$$T^* g(S) = g(TS), \tag{1.21}$$

for g satisfying (1.20). Note that if β is a section of L_k over H and $f : H \to \mathbb{C}$, then $f\beta$ is a section of L_k, and, for $T \in S\ell(2,\mathbb{R})$,

$$T^*(f\beta) = f(Tz) T^*\beta. \tag{1.22}$$

2. The disk model

It is convenient to produce formulas similar to those of §1, with the upper half plane H replaced by the disk $D = \{z \in \mathbb{C} : |z| < 1\}$. This is holomorphically equivalent to H via the conformal diffeomorphisms

$$\varphi : H \to D, \quad \psi = \varphi^{-1} : D \to H, \quad \varphi(z) = \frac{z-i}{z+i}, \quad \psi(z) = \frac{1}{i}\,\frac{z+1}{z-1}, \tag{2.1}$$

and the Poincaré metric (1.14) on H pulls back to

$$ds_D = \psi^*\, ds_H = \frac{2}{1-|z|^2}\, |dz|. \tag{2.2}$$

The group

$$SU(1,1) = \Bigl\{ T_{ab} = \begin{pmatrix} a & b \\ \overline{b} & \overline{a} \end{pmatrix} : a, b \in \mathbb{C},\ |a|^2 - |b|^2 = 1 \Bigr\} \tag{2.3}$$

acts on D by linear fractional transformations,

$$T_{ab}(z) = \frac{az+b}{\overline{b}z + \overline{a}}. \tag{2.4}$$

In this setting we take $\Gamma \subset SU(1,1)$ to be a discrete subgroup, and consider modular forms on D for Γ. As in (1.2), a modular form of weight 0 is just a holomorphic function on D (satisfying appropriate bounds) that satisfies

$$f(Tz) = f(z), \quad \forall\, T \in \Gamma,\ z \in D. \tag{2.5}$$

Next, parallel to (1.3), we look at holomorphic differential forms on D,

$$f(z)\, dz. \tag{2.6}$$

Note that dz on D and dz on H are related by the pull-back identity

$$\psi^*\, dz = \frac{2i}{(z-1)^2}\, dz. \tag{2.7}$$

Now, parallel to (1.5)–(1.6), we have

$$T_{ab}^*\, dz = T_{ab}'(z)\, dz, \quad T_{ab}'(z) = \frac{1}{(\overline{b}z + \overline{a})^2}, \tag{2.8}$$

so $T^*_{ab}(f\,dz) = f\,dz$ if and only if

$$f\Big(\frac{az+b}{\bar{b}z+\bar{a}}\Big) = (\bar{b}z+\bar{a})^2\, f(z), \quad \forall\, T_{ab} \in \Gamma. \tag{2.9}$$

If in addition f satisfies a growth condition parallel to that of §1, we say $f\,dz$ (or, briefly, f) is a modular form of weight 2.

Computations parallel to those of (1.8)–(1.11) hold, yielding the functional equation

$$f\Big(\frac{az+b}{\bar{b}z+\bar{a}}\Big) = (\bar{b}z+\bar{a})^{2k} f(z), \quad \forall\, T_{ab} \in \Gamma, \tag{2.10}$$

again supplemented by appropriate growth conditions, to define which holomorphic functions f on D give rise to modular forms of weight $2k$, $k \in \mathbb{Z}$.

By (2.7), a holomorphic function $f : H \to \mathbb{C}$ is a modular form of weight $2k$ for a discrete subgroup $\Gamma_h \subset S\ell(2,\mathbb{R})$ is and only if

$$(z-1)^{2k} f \circ \psi : D \longrightarrow \mathbb{C} \tag{2.11}$$

is a modular form of weight $2k$ for the subgroup $\Gamma_d \subset SU(1,1)$, conjugate to Γ_h via (2.1).

3. Compact quotients

If M is a compact Riemann surface of genus $g \geq 2$, then it has a compatible metric of curvature -1, so its universal covering surface is isometrically equivalent to the upper half plane H, with its Poincaré metric, and we have

$$M \approx \Gamma \setminus H, \tag{3.1}$$

where $\Gamma \subset S\ell(2,\mathbb{R})$ is the group of covering transformations of $H \to M$. See [T3], Theorem 2.1. In this case, the space of modular forms of weight k is equal to

$$\mathcal{O}(L_k), \tag{3.1}$$

where L_k is the complex line bundle described around (1.19). Here we record some well known facts about these spaces, including their dimensions. Notation will be as in [T2].

First, we have

$$L_k = L_1^k = L_1 \otimes \cdots \otimes L_1 (k \text{ factors}), \tag{3.3}$$

for $k \in \mathbb{N}$, and

$$L_2 = \kappa, \tag{3.4}$$

the canonical bundle. We have $c_1(\kappa)[M] = 2g - 2$ for the Chern class of κ, hence

$$c_1(L_k)[M] = k(g-1). \tag{3.5}$$

This allows a computation of $\dim \mathcal{O}(L_k)$ for $k \geq 2$, as a consequence of the following:

$$\dim \mathcal{O}(\kappa) = g, \tag{3.6}$$

and, for a holomorphic line bundle $L \to M$,

$$c_1(L)[M] > 2g-2 \Longrightarrow \dim \mathcal{O}(L) = c_1(L)[M] - (g-1). \tag{3.7}$$

See (9.26) of [T2] for (3.6) and (9.34) of [T2] for (3.7). The former result is elementary. (See [T1], (10.25)–(10.28).) The latter result is a paradigmatic consequence of the Riemann-Roch formula, which says that, for a general holomorphic line bundle L over a compact Riemann surface M, of genus g,

$$\dim \mathcal{O}(L) - \dim \mathcal{O}(L^{-1} \otimes \kappa) = c_1(L)[M] - (g-1). \tag{3.7A}$$

Consequently, for L_k as in (3.3), $g \geq 2$,

$$\dim \mathcal{O}(L_2) = g, \quad \text{and} \quad k \geq 3 \Rightarrow \dim \mathcal{O}(L_k) = (k-1)(g-1). \tag{3.8}$$

We also have

$$c_1(L)[M] < 0 \Longrightarrow \mathcal{O}(L) = 0, \tag{3.9}$$

see [T2], Corollary 9.4, hence

$$k < 0 \Longrightarrow \mathcal{O}(L_k) = 0. \tag{3.10}$$

Of course, $\dim \mathcal{O}(L_0) = 1$.

There remains the question of what is

$$\dim \mathcal{O}(L_1). \tag{3.11}$$

The following result gives some limited information. Generally, by (9.46) of [T2], if $L \to M$ is a holomorphic line bundle,

$$c_1(L)[M] = g - 1 \Longrightarrow 0 \leq \dim \mathcal{O}(L) \leq g. \tag{3.12}$$

As indicated in §1, our principal interest is in subgroups Γ somewhat different from those arising as covering transformations of $H \to M$ for a compact Riemann surface. The case $\Gamma = S\ell(2, \mathbb{Z})$ will be examined more thoroughly in §4. In this case, $\Gamma \setminus H$ is not compact, rather it is homeomorphic to $\widehat{\mathbb{C}} \setminus \{\infty\} = \mathbb{C}$. Also some elements of Γ have fixed points in H, so the map $H \to \mathbb{C}$ has branch points. Now $\mathbb{C}$ can be compactified to $\widehat{\mathbb{C}}$, and these branch points can be desingularized. In such a case, modular forms of weight $2k$ give rise to sections of κ^k on $\widehat{\mathbb{C}}$, but they are meromorphic, with poles. It is hence useful to recall extensions of the Riemann-Roch formula (3.7A) to deal with spaces of meromorphic sections of various holomorphic line bundles over a compact Riemann surface M, of genus g. We take the space here to set down needed results.

A *divisor* on M is a formal finite sum

$$\vartheta = \sum_p \nu(p) p, \quad \nu(p) \in \mathbb{Z}. \tag{3.13}$$

We define the order of the divisor to be $o(\vartheta) = \sum \nu(p)$. If $L \to M$ is a holomorphic line bundle and f is a meromorphic section of L, we set

$$\vartheta(f) = \sum_p \nu_f(p) p, \tag{3.14}$$

where $\nu_f(p) = k$ if p is a pole of f or order k, $\nu_f(p) = -\ell$ if p is a zero of f of order ℓ. We define

$$\mathcal{M}(L, \vartheta) \tag{3.15}$$

to be the space of meromorphic sections of L such that

$$\vartheta(f) \leq \vartheta, \tag{3.16}$$

that is, if ϑ is as in (3.13), then $\nu_f(p) \leq \nu(p)$ for all p. It is an important result (see Proposition 9.6 in [T2]) that to each divisor ϑ on M there corresponds a holomorphic line bundle $E_\vartheta \to M$ having the property that there is a natural isomorphism

$$\mathcal{M}(L, \vartheta) \approx \mathcal{O}(L \otimes E_\vartheta). \tag{3.17}$$

Furthermore,

$$c_1(L \otimes E_\vartheta)[M] = c_1(L)[M] + o(\vartheta). \tag{3.18}$$

Hence, from (3.7A) we have the following:

$$\dim \mathcal{M}(L, \vartheta) - \dim \mathcal{O}((L \otimes E_\vartheta)^{-1} \otimes \kappa) = c_1(L)[M] + o(\vartheta) - (g - 1). \tag{3.19}$$

In particular (cf. [T2], (9.42))

$$c_1(L)[M] + o(\vartheta) > 2g - 2 \Rightarrow \dim \mathcal{M}(L, \vartheta) = c_1(L)[M] + o(\vartheta) - (g - 1). \tag{3.20}$$

4. The modular group $S\ell(2,\mathbb{Z})$

The group $S\ell(2,\mathbb{Z})\subset S\ell(2,\mathbb{R})$ was mentioned in §1 as one of the primary discrete subgroups Γ of interest in the study of modular forms. For $\Gamma=S\ell(2,\mathbb{Z})$, the quotient $\Gamma\setminus H$ is not compact, but one has the following significant result.

Proposition 4.1. *For $\Gamma=S\ell(2,\mathbb{Z})$, the quotient $\Gamma\setminus H$ has the natural structure of a Riemann surface conformally equivalent to $\mathbb{C}$.*

See [G], p. 7. This holomorphic diffeomorphism

$$S\ell(2,\mathbb{Z})\setminus H\approx\mathbb{C} \tag{4.1}$$

lifts to a holomorphic map

$$\Phi:H\longrightarrow\mathbb{C}, \tag{4.2}$$

satisfying

$$\Phi\Bigl(\frac{az+b}{cz+d}\Bigr)=\Phi(z),\quad\forall\begin{pmatrix}a&b\\c&d\end{pmatrix}\in S\ell(2,\mathbb{Z}). \tag{4.3}$$

This is an infinite branched covering, with branch points of order 2 at $p_0=i$ and its images under Γ, and branch points of order 3 at $q_0=1/2+\sqrt{3}i/2$ and its images under Γ. We can arrange that $\Phi(p_0)=0$ and $\Phi(q_0)=1$. We have

$$\Phi(x+iy)\to\infty,\quad\text{as}\ \ y\nearrow+\infty, \tag{4.4}$$

uniformly in x (since $\Phi(z+1)=\Phi(z)$). These conditions imply that, for some $A\in(0,\infty)$,

$$\Phi(z)=\varphi(e^{2\pi iz}),\quad\text{for}\ \ \operatorname{Im}z\ge A, \tag{4.5}$$

where φ is meromorphic on the disk $\{\zeta\in\mathbb{C}:|\zeta|<e^{-2\pi A}\}$, with one simple pole, at $\zeta=0$, hence

$$\Phi(z)=\alpha e^{-2\pi iz}+O(1),\quad\text{for}\ \ \operatorname{Im}z>A. \tag{4.6}$$

Thus Φ does not count as a modular form of weight 0.

The following is a useful result about $S\ell(2,\mathbb{Z})$. See [T5] for a proof.

Proposition 4.2. *The group $S\ell(2,\mathbb{Z})$ is generated by the two elements*

$$S_1=\begin{pmatrix}1&1\\0&1\end{pmatrix}\ \ \textit{and}\ \ S_2=\begin{pmatrix}0&-1\\1&0\end{pmatrix}. \tag{4.7}$$

Corollary 4.3. *Given $k \in \mathbb{Z}$, a holomorphic function $f : H \to \mathbb{C}$ satisfies*

$$f\Big(\frac{az+b}{cz+d}\Big) = (cz+d)^{2k} f(z), \quad \forall\, z \in H,\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in S\ell(2,\mathbb{Z}), \tag{4.8}$$

if and only if

$$f(z+1) = f(z), \quad \text{and} \ \ f\Big(-\frac{1}{z}\Big) = z^{2k} f(z), \quad \forall\, z \in H. \tag{4.9}$$

Proof. Condition (4.8) says

$$T^*(f\,(dz)^k) = f\,(dz)^k \tag{4.10}$$

for all $T \in S\ell(2,\mathbb{Z})$, and (4.9) says this holds for $T = S_1$ and $T = S_2$.

The designation of $S\ell(2,\mathbb{Z})$ as "the modular group" refers to its role in the description of the moduli space of the collection of elliptic curves, i.e., the compact Riemann surfaces of genus $g = 1$. Any such Riemann surface is holomorphically equivalent to a torus

$$\mathbb{T}_\Lambda = \mathbb{C}/\Lambda, \tag{4.11}$$

for some lattice $\Lambda \subset \mathbb{C}$. See [T2], Proposition 9.10. Two such tori $\mathbb{T}_\Lambda$ and $\mathbb{T}_{\Lambda'}$ are holomorphically equivalent if and only if

$$\Lambda' = \lambda\Lambda, \ \text{for some } \lambda \in \mathbb{C}^* = \mathbb{C} \setminus 0. \tag{4.12}$$

Let $\mathcal{L}$ denote the space of lattices in $\mathbb{C}$, and let

$$M = \Big\{ \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} \in \mathbb{C}^2 : \operatorname{Im} \frac{\omega_1}{\omega_2} > 0 \Big\}. \tag{4.13}$$

The map $M \to \mathcal{L}$ given by $(\omega_1, \omega_2)^t \mapsto \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ is surjective, and two such pairs $(\omega_1, \omega_2)^t$ and $(\eta_1, \eta_2)^t$ define the same lattice if and only if

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}, \quad \text{for some } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in S\ell(2,\mathbb{Z}). \tag{4.14}$$

This identifies $\mathcal{L} \approx \Gamma \setminus M$. Note that

$$\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 \Rightarrow \omega_2^{-1}\Lambda = \mathbb{Z}\tau + \mathbb{Z}, \quad \tau = \frac{\omega_1}{\omega_2} \in H. \tag{4.15}$$

If also $\Lambda' = \mathbb{Z}\eta_1 + \mathbb{Z}\eta_2$, yielding $\eta_2^{-1}\Lambda' = \mathbb{Z}\sigma + \mathbb{Z}$, $\sigma = \eta_1/\eta_2 \in H$, then the relation (4.14) implies

$$\frac{a\sigma + b}{c\sigma + d} = \tau. \tag{4.16}$$

Hence we have a natural equivalence

$$\mathcal{L}/\mathbb{C}^* \approx \Gamma \setminus H, \quad \Gamma = S\ell(2,\mathbb{Z}), \tag{4.17}$$

defining the moduli space of elliptic curves.

Using (4.13)–(4.17), we can provide another perspective on the space of modular forms. We follow [V], pp. 242–243. We say a function $F : \mathcal{L} \to \mathbb{C}$ is of weight $2k$ provided

$$F(\lambda\Lambda) = \lambda^{-2k} F(\Lambda), \quad \Lambda \in \mathcal{L},\ \lambda \in \mathbb{C} \setminus 0. \tag{4.18}$$

(Note that if we replace $2k$ by an odd integer, the analogous identity forces $F \equiv 0$, since $\Lambda = -\Lambda$.) If we let $\Lambda_\tau = \mathbb{Z}\tau + \mathbb{Z}$ and set $f(\tau) = F(\Lambda_\tau)$, then we see the holomorphic lattice functions of weight $2k$ are equivalent to the holomorphic functions $f : H \to \mathbb{C}$ satisfying (4.8).

We now describe how a family of modular forms arises from the study of certain elliptic functions, namely the Weierstrass $\wp$-functions

$$\wp_\Lambda(z) = \frac{1}{z^2} + \sum_{\omega \in \Lambda \setminus 0} \Bigl(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \Bigr). \tag{4.19}$$

This has a pole at $z = 0$, and we have

$$\Phi_\Lambda(z) = \wp_\Lambda(z) - \frac{1}{z^2} \tag{4.20}$$

holomorphic on a neighborhood of $z = 0$. A calculation gives $\Phi_\Lambda(0) = 0$ and, for $k \ge 1$,

$$\frac{1}{k!} \Phi_\Lambda^{(k)}(0) = (k+1) \sum_{\omega \in \Lambda \setminus 0} \omega^{-(k+2)}. \tag{4.21}$$

See [T4], §6.2. These coefficients vanish for k odd. The sums

$$G_k(\Lambda) = \sum_{\omega \in \Lambda \setminus 0} \omega^{-2k}, \quad k \ge 2, \tag{4.22}$$

are known as Eisenstein series. Clearly

$$G_k(\lambda\Lambda) = \lambda^{-2k} G_k(\Lambda), \tag{4.23}$$

as in (4.18). We deduce that the functions

$$G_k(\tau) = \sum_{(m,n) \ne (0,0)} (m\tau + n)^{-2k} \tag{4.24}$$

define modular forms of weight $2k$, for $k \ge 2$.

These Eisenstein series are related to other facets of the behavior of $\wp_\Lambda$. For example, if $\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ and

$$e_j(\Lambda) = \wp_\Lambda\Bigl(\frac{\omega_j}{2}\Bigr), \quad j = 1, 2, 3,\ \omega_3 = \omega_1 + \omega_2, \tag{4.25}$$

then ([T4], (6.2.7))

$$\wp_\Lambda'(z)^2 = 4(\wp_\Lambda(z) - e_1)(\wp_\Lambda(z) - e_2)(\wp_\Lambda(z) - e_3), \tag{4.26}$$

and furthermore ([T4], (6.2.15), (6.2.27))

$$\begin{aligned} e_1 + e_2 + e_3 &= 0, \\ e_1e_2 + e_2e_3 + e_1e_3 &= -15G_2(\Lambda), \\ e_1e_2e_3 &= 35G_3(\Lambda). \end{aligned} \tag{4.27}$$

In addition, with

$$b_n = \frac{1}{(2n)!}\Phi_\Lambda^{(2n)}(0) = (2n+1)G_{n+1}(\Lambda), \tag{4.28}$$

we have ([T4], (6.2.24)) a recursive formula for the sequence (b_n), given b_2, b_3, as can be deduced from the differential equation ([T4], (6.2.22))

$$\wp_\Lambda''(z) = 6\wp_\Lambda(z)^2 - 30G_2(\Lambda). \tag{4.29}$$

Hence, for $n \ge 4$,

$$G_n(\Lambda) = P_n(G_2(\Lambda), G_3(\Lambda)), \tag{4.30}$$

where $P_n(G_2, G_3)$ is a polynomial in G_2, G_3, with coefficients that are positive rational numbers.

5. Dimension of spaces of modular forms for $\Gamma = S\ell(2,\mathbb{Z})$

As mentioned in §4, when $\Gamma = S\ell(2,\mathbb{Z})$, we have a holomorphic diffeomorphism $\Gamma \setminus H \approx \mathbb{C}$, lifting to a holomorphic map $\Phi : H \to \mathbb{C}$, satisfying (4.3), with branch points of order 2 at $p_0 = i$ and its images under Γ and branch points of order 3 at $q_0 = 1/2 + \sqrt{3}i/2$ and its images under Γ, and we can arrange that $\Phi(p_0) = 0$, $\Phi(q_0) = 1$. Also $\Phi(z) \to \infty$ as $\operatorname{Im} z \nearrow \infty$. We now examine how a modular form f of weight $2k$ on H yields a meromorphic section of κ^k on $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, perhaps with poles at $\infty, 0$, and 1.

For the cusp at ∞ in H, using $\Phi(z+1) = \Phi(z)$, we set

$$w = e^{2\pi i z}, \quad \frac{dw}{w} = 2\pi i \, dz. \tag{5.1}$$

If $f\,(dz)^k$ is a modular form of weight $2k$, we set $f(z) = g(e^{2\pi i z})$. Then

$$f(z)\,(dz)^k = (2\pi i)^{-k} \frac{g(w)}{w^k}\,(dw)^k. \tag{5.2}$$

Hence the section φ of κ^k over $\widehat{\mathbb{C}}$ associated to $f\,(dz)^k$ has a pole of order k at ∞.

To examine the behavior of such φ at $0 \in \widehat{\mathbb{C}}$, it is convenient apply a linear fractional transformation to H, taking i to 0, so the 2-fold branch cover by H at p_0 is converted to

$$w = z^2, \quad dw = 2z\,dz. \tag{5.3}$$

If $f\,(dz)^k$ is a modular form of weight $2k$, we have

$$f(-z) = (-1)^k f(z), \tag{5.4}$$

hence

$$\begin{aligned} f(z) &= z g(z^2), \quad k \text{ odd}, \\ f(z) &= g(z^2), \quad k \text{ even}. \end{aligned} \tag{5.5}$$

This leads to the following computations:

$$\begin{aligned} \text{weight } 2: \ & f(z)\,(dz) = g(z^2) z\,dz = \frac{g(w)}{2}\,dw, \\ \text{weight } 4: \ & f(z)\,(dz)^2 = g(z^2)\,(dz)^2 = \frac{g(w)}{4w}\,(dw)^2, \\ \text{weight } 6: \ & f(z)\,(dz)^3 = g(z^2) z\,(dz)^3 = \frac{g(w)}{8w}\,(dw)^3. \end{aligned} \tag{5.6}$$

Further weights $2k$ lead to analogous formulas.

To examine the behavior of φ at $1 \in \widehat{\mathbb{C}}$, it is convenient to shift by -1, converting 1 to 0, and to apply a linear fractional transformation to H, taking q_0 to 0, so the 3-fold branched cover by H at q_0 is converted to

$$w = z^3, \quad dw = 3z^2\, dz. \tag{5.7}$$

If $f\,(dz)^k$ is a modular form of weight $2k$, we have

$$f(\omega z) = \omega^{-k} f(z) = \omega^{2k} f(z), \quad \omega = e^{2\pi i/3}, \tag{5.8}$$

hence

$$\begin{aligned} f(z) &= z^2 g(z^3), \quad k = 1, \\ f(z) &= z g(z^3), \quad k = 2, \\ f(z) &= g(z^3), \quad k = 3, \end{aligned} \tag{5.9}$$

and so on. This leads to the following computations:

$$\begin{aligned} \text{weight } 2: \ f(z)\,(dz) &= g(z^3) z^2\, dz = \frac{g(w)}{3}\, dw, \\ \text{weight } 4: \ f(z)\,(dz)^2 &= g(z^3) z\,(dz)^2 = \frac{g(w)}{9w}\,(dw)^2, \\ \text{weight } 6: \ f(z)\,(dz)^3 &= g(z^3)\,(dz)^3 = \frac{g(w)}{27w^2}\,(dw)^3, \end{aligned} \tag{5.10}$$

and so on.

Thus, if $f\,(dz)^k$ is a modular form of weight $2k$ on H (for Γ), we can say the following about the associated meromorphic section φ of κ^k on $\widehat{\mathbb{C}}$:

(5.11)

$k = 1$: simple pole at ∞, regular at 0, regular at 1,
$k = 2$: double pole at ∞, simple pole at 0, simple pole at 1,
$k = 3$: triple pole at ∞, simple pole at 0, double pole at 1.

Hence, in these cases, we have

$$\varphi \in \mathcal{M}(\kappa^k, \vartheta_k), \tag{5.12}$$

with

$$\begin{aligned} \vartheta_1 &= 1\{\infty\} + 0\{0\} + 0\{1\}, \quad o(\vartheta_1) = 1, \\ \vartheta_2 &= 2\{\infty\} + 1\{0\} + 1\{1\}, \quad o(\vartheta_2) = 4, \\ \vartheta_3 &= 3\{\infty\} + 1\{0\} + 2\{1\}, \quad o(\vartheta_3) = 6. \end{aligned} \tag{5.13}$$

Recall from (3.17) that

$$\mathcal{M}(\kappa^k, \vartheta_k) \approx \mathcal{O}(\kappa^k \otimes E_{\vartheta_k}), \tag{5.14}$$

where $E_{\vartheta_k} \to \widehat{\mathbb{C}}$ is a holomorphic line bundle, and, by (3.18),

$$c_1(\kappa^k \otimes E_{\vartheta_k})[\widehat{\mathbb{C}}] = 2k(g-1) + o(\vartheta_k) = -2k + o(\vartheta_k), \tag{5.15}$$

since $\widehat{\mathbb{C}}$ has genus $g = 0$. In particular,

$$k = 1 \Rightarrow c_1(\kappa^k \otimes E_{\vartheta_k})[\widehat{\mathbb{C}}] = -1 \Rightarrow \mathcal{M}(\kappa^k, \vartheta_k) = 0, \tag{5.16}$$

so there are no nonzero modular forms of weight 2 for $S\ell(2,\mathbb{Z})$. Next,

$$\begin{aligned} k = 2 &\Rightarrow c_1(\kappa^k \otimes E_{\vartheta_k})[\widehat{\mathbb{C}}] = -4 + 4 = 0, \\ k = 3 &\Rightarrow c_1(\kappa^k \otimes E_{\vartheta_k})[\widehat{\mathbb{C}}] = -6 + 6 = 0. \end{aligned} \tag{5.17}$$

The following consequence of (3.7A) is incisive.

Proposition 5.1. *If $L \to \widehat{\mathbb{C}}$ is a holomorphic line bundle, then*

$$c_1(L)[\widehat{\mathbb{C}}] = 0 \Longrightarrow \dim \mathcal{O}(L) = 1. \tag{5.18}$$

In this case, each nontrivial $f \in \mathcal{O}(L)$ is nowhere vanishing, so L is holomorphically trivial.

Corollary 5.2. *For $\Gamma = S\ell(2,\mathbb{Z})$, the space of modular forms of weight 4 and of weight 6 are both one dimensional. Hence these spaces are, respectively, spanned by the Eisenstein series G_2 and G_3, given by (4.24).*

For general k, we can say that

$$\begin{aligned} o(\vartheta_k) < 2k &\Rightarrow \mathcal{M}(\kappa^k, \vartheta_k) = 0, \\ o(\vartheta_k) \ge 2k &\Rightarrow \dim \mathcal{M}(\kappa^k, \vartheta_k) = o(\vartheta_k) - 2k + 1. \end{aligned} \tag{5.19}$$

by (5.15) and (3.7A). Pursuing computations along the lines of (5.6)–(5.13), one obtains the following (cf. [G], p. 26):

$$\begin{aligned} D_k = \dim \mathcal{M}(\kappa^k, \vartheta_k) = &\left[\frac{k}{6}\right], \qquad \text{if } k = 1 \bmod 6, \\ &\left[\frac{k}{6}\right] + 1, \text{ if } k \ne 1 \bmod 6, \end{aligned} \tag{5.20}$$

where $[x]$ denotes the greatest integer $\le x$. For example, $D_1 = 0$ (as seen above), and, extending Corollary 5.2,

$$\begin{aligned} D_k = &\ 1 \text{ for } k \in \{2,3,4,5,7\}, \\ &\ 2 \text{ for } k \in \{6,8,9,10,11,13\}. \end{aligned} \tag{5.21}$$

The implication for modular forms is the following.

Proposition 5.3. *Let $\mathcal{M}_k$ denote the space of modular forms of weight k for $\Gamma = S\ell(2,\mathbb{Z})$. Then*

$$\dim \mathcal{M}_{2k} = D_k, \tag{5.22}$$

given by (5.20).

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