Orthogonal Projections onto Ranges and Null Spaces

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Assume we have linear transformations

$$\begin{aligned} A &: \mathbb{R}^k \longrightarrow \mathbb{R}^n, \quad \text{injective}, \\ B &: \mathbb{R}^n \longrightarrow \mathbb{R}^k, \quad \text{surjective}. \end{aligned} \tag{1}$$

We seek formulas for the orthogonal projections P and Q on $\mathbb{R}^n$, defined by

$$\begin{aligned} P &= \perp \text{ projection of } \mathbb{R}^n \text{ onto } \mathcal{R}(A), \\ Q &= \perp \text{ projection of } \mathbb{R}^n \text{ onto } \mathcal{N}(B). \end{aligned} \tag{2}$$

To start, note that

$$\mathcal{N}(A^t) = \mathcal{R}(A)^\perp, \quad A^tA : \mathbb{R}^k \xrightarrow{\approx} \mathbb{R}^k, \tag{3}$$

This leads to the following result.

Proposition 1. *The orthogonal projection P in (2) is given by*

$$P = A(A^tA)^{-1}A^t. \tag{4}$$

Proof. Clearly $P^t = P$. (It is also routine to calculate that $P^2 = P$.) Next,

$$v \perp \mathcal{R}(A) \Longrightarrow A^t v = 0 \Longrightarrow Pv = 0. \tag{5}$$

Finally,

$$v = Au \ (u \in \mathbb{R}^k) \Longrightarrow Pv = A(A^tA)^{-1}A^tAu = Au = v. \tag{6}$$

This proves Proposition 1.

Moving on to the calculation of Q, we have

$$BB^t : \mathbb{R}^k \xrightarrow{\approx} \mathbb{R}^k, \quad \mathcal{R}(B^t) = \mathcal{N}(B)^\perp. \tag{7}$$

This leads to the following formula for $Q^\perp = I - Q$.

Proposition 2. *For Q as in (2),*

$$Q^\perp = B^t(BB^t)^{-1}B. \tag{8}$$

Proof. Again clearly $(Q^\perp)^t = Q^\perp$ (and a calculation gives $(Q^\perp)^2 = Q^\perp$). Next

$$v \in \mathcal{N}(B) \Longrightarrow Bv = 0 \Longrightarrow Q^\perp v = 0. \tag{9}$$

Finally,

$$\begin{aligned} v \perp \mathcal{N}(B) &\Longrightarrow v = B^t u, \ \text{for some } u \in \mathbb{R}^k \\ &\Longrightarrow Q^\perp v = B^t(BB^t)^{-1}BB^tu = B^tu = v. \end{aligned} \tag{10}$$

This proves Proposition 2.