Heron's Formula

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Let $\mathcal{T} \subset \mathbb{R}^2$ be a triangle, with vertices A, B, C, having opposing sides of length a, b, c, respectively. Set

$$s = \frac{1}{2}(a + b + c). \tag{1}$$

Heron's formula for the area of $\mathcal{T}$ is the following

$$(\text{Area}\,\mathcal{T})^2 = s(s-a)(s-b)(s-c). \tag{2}$$

To set up a proof, let us note that at most one angle of $\mathcal{T}$ is obtuse. We can assume A and B are acute. Drop a perpendicular from C to the line segment from A to B, dividing this segment into two pieces, of length x and y, so

$$x + y = c. \tag{3}$$

See Figure 1. Let h denote the length of this perpendicular line segment. Hence

$$\text{Area}\,\mathcal{T} = \frac{1}{2}ch. \tag{4}$$

Meanwhile, the Pythagorian theorem yields

$$h^2 + x^2 = b^2, \quad h^2 + y^2 = a^2. \tag{5}$$

Our task is to deduce (2) from (3)–(5).

Subtracting in (5) gives $x^2 - y^2 = b^2 - a^2$, and since

$$x^2 - y^2 = (x+y)(x-y) = c(x-y),$$

we have

$$x - y = \frac{b^2 - a^2}{c}. \tag{6}$$

Adding (3) and (6) gives

$$x = \frac{c^2 + b^2 - a^2}{2c}, \tag{7}$$

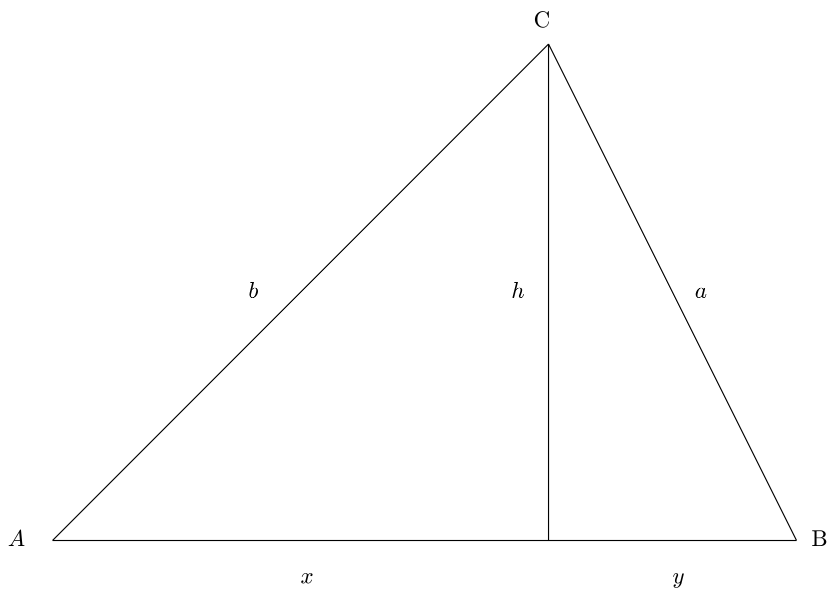

Figure 1: Triangle

and hence from (5) we get

$$h^2 = b^2 - x^2 = b^2 - \frac{(c^2 + b^2 - a^2)^2}{4c^2}. \tag{8}$$

Therefore,

$$\begin{aligned} 4c^2h^2 &= (2bc)^2 - (c^2 + b^2 - a^2)^2 \\ &= (2bc + c^2 + b^2 - a^2)(2bc - c^2 - b^2 + a^2) \\ &= [(b + c)^2 - a^2] \cdot [a^2 - (b - c)^2] \\ &= (a + b + c)(-a + b + c)(a + b - c)(a - b + c) \\ &= 16s(s - a)(s - b)(s - c). \end{aligned} \tag{9}$$

By (4),

$$(\text{Area}\,\mathcal{T})^2 = \frac{c^2h^2}{4}, \tag{10}$$

and we have the desired identity (2).