

Topological Obstruction to Lifting Principal Bundles

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Let G be a connected Lie group with connected covering group \tilde{G} , and covering homomorphism $\pi : \tilde{G} \rightarrow G$. In such a case, the discrete group $\Gamma = \text{Ker } \pi$ is contained in the center of \tilde{G} . The basic case is $Spin(n) \rightarrow SO(n)$.

Let M be a Riemannian manifold and $P \rightarrow M$ a principal G -bundle. Cover M with geodesically convex balls U_j . Then P is trivializable over each U_j , say $P \approx U_j \times G$. If $x \in U_{jk} = U_j \cap U_k$, the two trivializations (over U_k and U_j , respectively) are related by

$$(1) \quad T_{jk}(g, x) = (g_{jk}(x)g, x), \quad g_{jk} \in C(U_{jk}, G), \quad g_{jk}^{-1} = g_{kj}.$$

It follows that if $U_{jkl} = U_j \cap U_k \cap U_l \neq \emptyset$ and $x \in U_{jkl}$, then

$$(2) \quad g_{jk} = g_{jl}g_{lk}.$$

The last identity can be rewritten

$$(3) \quad g_{jk}g_{kl}g_{lj} \equiv e,$$

where $e \in G$ is the identity element.

We want to see when it is possible to lift P to a principal \tilde{G} -bundle $\tilde{P} \rightarrow M$. As a first try, we can take

$$(4) \quad \tilde{g}_{jk} \in C(U_{jk}, \tilde{G}) \quad \text{such that} \quad \pi \circ \tilde{g}_{jk} = g_{jk}.$$

This will define a principal \tilde{G} -bundle provided that the analogue $\tilde{g}_{jk}\tilde{g}_{kl}\tilde{g}_{lj} = e$ of (3) holds, but we cannot expect that. We do know that

$$(5) \quad \pi(\tilde{g}_{jk}\tilde{g}_{kl}\tilde{g}_{lj}) = e, \quad \text{hence} \quad \tilde{g}_{jk}\tilde{g}_{kl}\tilde{g}_{lj} = e_{jkl} \in \Gamma,$$

whenever $U_j \cap U_k \cap U_l \neq \emptyset$. We next try to adjust the local lifts \tilde{g}_{jk} . We set

$$(6) \quad g_{jk}^\#(x) = f_{jk}\tilde{g}_{jk}(x), \quad f_{jk} \in \Gamma.$$

Recalling that Γ is in the center of \tilde{G} , we see that

$$(7) \quad \begin{aligned} g_{jk}^\#g_{kl}^\#g_{lj}^\# &= f_{jk}f_{kl}f_{lj}\tilde{g}_{jk}\tilde{g}_{kl}\tilde{g}_{lj} \\ &= f_{jk}f_{kl}f_{lj}e_{jkl}. \end{aligned}$$

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Hence $\{g_{jk}^\# \in C(U_{jk}, \tilde{G})\}$ defines a lift of P if and only if

$$(8) \quad e_{jkl} = \varphi_{jk}\varphi_{kl}\varphi_{lj}, \quad \varphi_{jk} = f_{jk}^{-1}.$$

Now (8) says that $\{e_{jkl}\}$ is a coboundary, hence defines the element 0 in the cohomology group $H^2(M, \Gamma)$. Let us also note the following. Assume the cover $\{U_j\}$ of M is sufficiently fine that the following holds. Given any 4 elements of this cover with connected union (say \mathcal{O}), then this union is contained in a geodesically convex set. Then P trivializes over \mathcal{O} , hence lifts. Consequently the restriction of $\{e_{jkl}\}$ to \mathcal{O} is a coboundary. It follows that $\{e_{jkl}\}$ defines a *cocycle*, hence a cohomology class in $H^2(M, \Gamma)$. The computation (7) shows that any two local lifts $\{\tilde{g}_{jk}\}$ and $\{g_{jk}^\#\}$ define cohomologous cocycles. This gives the following result.

Proposition 1. *Associated to the principal G -bundle $P \rightarrow M$ and connected covering group $\tilde{G} \rightarrow G$ is a cohomology class*

$$(9) \quad w(P, \tilde{G}) \in H^2(M, \Gamma).$$

Furthermore, P can be lifted to a principal \tilde{G} -bundle $\tilde{P} \rightarrow M$ if and only if $w(P, \tilde{G}) = 0$.

Suppose now that $w(P, \tilde{G}) = 0$ and that $\{\tilde{g}_{jk}\}$ and $\{g_{jk}^\#\}$ both lift P to principal \tilde{G} -bundles, so $\pi(\tilde{g}_{jk}) = \pi(g_{jk}^\#) = g_{jk}$ on U_{jk} , and

$$(10) \quad \tilde{g}_{jk}\tilde{g}_{kl}\tilde{g}_{lj} \equiv e, \quad g_{jk}^\#g_{kl}^\#g_{lj}^\# \equiv e,$$

where here e denotes the identity on \tilde{G} . Again, \tilde{g}_{jk} and $g_{jk}^\#$ are related as in (6), and the identities (10) then imply that

$$(11) \quad f_{jk}f_{kl}f_{lj} \equiv e.$$

That is to say, $\{f_{jk}\}$ is a cocycle, defining an element

$$(12) \quad [\{f_{jk}\}] \in H^1(M, \Gamma).$$

Furthermore, if $\{\tilde{g}_{jk}\}$ satisfies (10) and $\{g_{jk}^\#\}$ are given by (6), then $\{g_{jk}^\#\}$ satisfies (10) if and only if $\{f_{jk}\}$ satisfies (11), hence if and only if we get an element $[\{f_{jk}\}] \in H^1(M, \Gamma)$. This element is zero if and only if there exist $e_j \in \Gamma$ such that

$$(13) \quad f_{jk} = e_j^{-1}e_k.$$

If (13) holds, then, by (6),

$$(14) \quad e_j g_{jk}^\#(x) = \tilde{g}_{jk}(x)e_k,$$

hence $\{e_k\}$ defines a bundle isomorphism $\tilde{P}_1 \rightarrow \tilde{P}_2$, where \tilde{P}_1 is the lift of P defined by $\{g_{jk}^\#\}$ and \tilde{P}_2 the lift defined by $\{\tilde{g}_{jk}\}$.

Examples of $\tilde{G} \rightarrow G$

1. $\text{Spin}(n) \rightarrow \text{SO}(n)$, $\text{SO}(n) \rightarrow \text{PSO}(n)$, n even.
2. $G \rightarrow \text{Ad } G$, (G connected Lie group with discrete center).
3. $Sl(2, \mathbb{R}) \rightarrow PSl(2, \mathbb{R})$.
4. $SU(n) \rightarrow SU(n)/\Gamma$, $\Gamma = \{\alpha I : \alpha^n = 1\}$.
5. $U(n) \rightarrow U(n)/\Gamma_m$, $\Gamma_m = \{\alpha I : \alpha^m = 1\}$.