Frenet Frames on Curves in Two and Three Dimensions

Michael Taylor

Abstract

We describe the Frenet frames (T, N) and (T, N, B), associated with smooth unit-speed curves γ, with tangent $T(s) = \gamma'(s)$, on $\mathbb{R}^2$, and $\mathbb{R}^3$, respectively, and derive the Frenet-Serret equations, involving also curvature κ and, for $n = 3$, torsion τ. The presentation in §1 requires T' to be nowhere vanishing when $n = 3$, and requires $\kappa > 0$. In §2 we allow T' to vanish simply at points s_j, and obtain smooth Frenet frames, with κ changing sign across each s_j. We extend this in §3 to cases where T' vanishes to finite order at s_j. In §4 we give examples where T' vanishes to infinite order and one cannot obtain a smooth, or even continuous Frenet frame.

1 Introduction

Let $\gamma : I \to \mathbb{R}^n$ be a smooth curve, where $I = (a, b) \subset \mathbb{R}$ is an interval. Assume the velocity vector $\gamma'(s)$ is nowhere vanishing, and in fact that γ is parametrized by arclength, so

$$T(s) = \gamma'(s) \Longrightarrow |T(s)| \equiv 1. \tag{1.1}$$

We call $T(s)$ the unit tangent field to γ. Note that γ is a straight line if and only if $T(s)$ is constant, so the derivative $T'(s)$ is a measure of the deviation of γ from being straight. One calls $T'(s)$ the curvature vector of γ. Note that

$$T(s) \cdot T(s) \equiv 1 \Rightarrow T'(s) \cdot T(s) \equiv 0 \Rightarrow T'(s) \perp T(s). \tag{1.2}$$

If $n = 2$, so $\gamma : I \to \mathbb{R}^2$ is a planar curve, then we can set

$$N(s) = JT(s), \tag{1.3}$$

where J denotes counterclockwise rotation on $\mathbb{R}^2$ by 90°, and call $N(s)$ the unit normal field to γ. In such a case, (1.2) implies $T'(s)$ is parallel to $N(s)$, so

$$T'(s) = \kappa(s)N(s), \tag{1.4}$$

for some smooth $\kappa : I \to \mathbb{R}$, called the (scalar) curvature of γ. Alternatively,

$$T'(s) = \kappa(s)JT(s). \tag{1.5}$$

Note that since (1.3) implies $N'(s) = JT'(s)$, applying J to (1.5) gives

$$N'(s) = \kappa(s)J^2T(s) = -\kappa(s)T(s). \tag{1.6}$$

In the planar case, we say $\{T(s), N(s)\}$ is a Frenet field for γ, and the equations (1.4) and (1.6) are called the Frenet equations. Alternatively, we can just say that (1.5) is the 2D Frenet equation.

It is elementary to solve the ODE (1.5), with $\kappa(s)$ given. In fact,

$$\begin{aligned}\kappa(s) = \alpha'(s) \Rightarrow \frac{d}{ds}e^{-\alpha(s)J} &= -\kappa(s)e^{-\alpha(s)J}J \\ \Rightarrow \frac{d}{ds}\Big(e^{-\alpha(s)J}T(s)\Big) &= e^{-\alpha(s)J}\Big(T'(s) - \kappa(s)JT(s)\Big),\end{aligned} \tag{1.7}$$

so (1.5) is equivalent to

$$e^{-\alpha(s)J}T(s) = e^{-\alpha(s_0)J}T_0, \tag{1.8}$$

for all $s, s_0 \in I$, with $T_0 = T(s_0)$. Since $J^t = -J$, we have

$$|e^{-\alpha(s)J}V| \equiv |V|, \tag{1.9}$$

so when (1.8) holds, $|T(s)| \equiv |T_0|$. Of course, having $T(s)$, one sets

$$\gamma(s) = \gamma(s_0) + \int_{s_0}^{s} T(t)\, dt. \tag{1.10}$$

If $n = 3$, then (1.2) continues to hold, but we no longer have a linear transformation like J to produce the normal $N(s)$. Instead, it is common to take

$$T'(s) = \kappa(s)N(s), \quad \kappa(s) = |T'(s)|, \quad N(s) = |T'(s)|^{-1}T'(s). \tag{1.11}$$

This gives well-defined, smooth functions κ and N, provided

$$T'(s) \neq 0, \quad \forall\, s \in I. \tag{1.12}$$

Note that with such definitions $\kappa(s) > 0$, as opposed to the 2D situation in (1.4), where $\kappa(s)$ can change sign. From (1.11) we can form

$$B(s) = T(s) \times N(s), \tag{1.13}$$

yielding an orthonormal frame, also satisfying

$$N(s) = B(s) \times T(s), \quad T(s) = N(s) \times B(s). \tag{1.14}$$

We seek formulas for the derivatives of N and B, complementing (1.11). Note that

$$\begin{aligned} N \cdot T \equiv 0 &\Rightarrow N' \cdot T + N \cdot T' = 0 \Rightarrow N' \cdot T = -\kappa, \\ N \cdot N \equiv 1 &\Rightarrow N' \cdot N = 0 \Rightarrow N' = -\kappa T + \tau B, \end{aligned} \tag{1.15}$$

where the last identity defines the function $\tau : I \to \mathbb{R}$, called the torsion of γ. Furthermore, differentiating (1.13) gives

$$\begin{aligned} B' &= T' \times N + T \times N' \\ &= 0 + T \times (-\kappa T + \tau B) \\ &= -\tau N, \end{aligned} \tag{1.16}$$

the last identity using (1.14). We collect these identities:

$$\begin{aligned} T' &= \qquad\quad \kappa N \\ N' &= -\kappa T \qquad\quad + \tau B \\ B' &= \qquad -\tau N \end{aligned} \tag{1.17}$$

These are called the Frenet-Serret equations for the frame field

$$F(s) = (T(s), N(s), B(s)). \tag{1.18}$$

This presents $F(s)$ as a 3×3 matrix. The statement that its columns form an orthonormal frame satisfying (1.13) is equivalent to the statement that $F(s)$ is a rotation matrix, i.e.,

$$F(s) \in SO(3). \tag{1.19}$$

The Frenet-Serret system (1.17) can be written in matrix form

$$F'(s) = F(s)A(s), \tag{1.20}$$

with

$$A(s) = \begin{pmatrix} 0 & -\kappa(s) & 0 \\ \kappa(s) & 0 & -\tau(s) \\ 0 & \tau(s) & 0 \end{pmatrix}. \tag{1.21}$$

The system (1.20) is solvable for arbitrarily given smooth functions $\kappa, \tau : I \to \mathbb{R}$. See §3.7 of [1] for details.

Now this last observation puts us in a curious situation. In taking arbitrary smooth functions $\kappa, \tau : I \to \mathbb{R}$, we could pick κ to vanish somewhere, and change sign. As long as $F_0 \in SO(3)$, the solution to (1.20) with initial data $F(s_0) = F_0$ will be an orthonormal frame field, and we can define $\gamma(s)$ as in (1.10) to be a smooth curve with "Frenet frame" given by $F(s)$. But the hypothesis (1.12) will be violated, and the formulas for $\kappa(s)$ and $N(s)$ are different from those written in (1.11).

Under these circumstances, we have strong motivation to define Frenet frames for a class of smooth, unit-speed curves $\gamma : I \to \mathbb{R}^3$ for which (1.1) holds but (1.12) is violated. We will take this up in §2, where T' is allowed to vanish simply at points $s_1, \dots, s_\ell \in I$. We extend this treatment in §3, allowing T' to vanish to finite order at the points s_j. In §4 we discuss counterexamples, showing that if we allow T' to vanish to infinite order at s_j, there might not exist a continuous normal field N.

2 Frenet frames for curves with simply vanishing curvature vector

We say a smooth curve $\gamma : I \to \mathbb{R}^3$ with tangent $T(s) = \gamma'(s)$ satisfying (1.1) has simply vanishing curvature vector at $s_0 \in I$ provided

$$T'(s_0) = 0, \quad T''(s_0) = V_0 \neq 0. \tag{2.1}$$

Here we show how to define smooth functions $\kappa : I_0 \to \mathbb{R},\ N : I_0 \to S^2$ such that

$$T'(s) = \kappa(s)N(s), \quad s \in I_0, \tag{2.2}$$

where I_0 is a neighborhood of s_0 in I. For notational simplicity we can take $s_0 = 0$. Then (2.1) says

$$T(s) = T_0 + \frac{s^2}{2}V(s), \quad V(0) = V_0 \neq 0, \tag{2.3}$$

where $V : I_0 \to \mathbb{R}^3$ is smooth and $|T_0| = 1$. Note that

$$\begin{aligned} &1 \equiv |T(s)|^2 = 1 + s^2 T_0 \cdot V(s) + \frac{s^4}{4}|V(s)|^2 \\ &\Longrightarrow T_0 \cdot V_0 = 0. \end{aligned} \tag{2.4}$$

Given (2.3),

$$T'(s) = sV(s) + \frac{s^2}{2}V'(s). \tag{2.5}$$

We now define κ and N on I_0 by

$$\kappa(s) = s\Big|V(s) + \frac{s}{2}V'(s)\Big| = s\varphi(s), \tag{2.6}$$

and

$$N(s) = \frac{1}{\kappa(s)}T'(s) = \frac{1}{\varphi(s)}\Big(V(s) + \frac{s}{2}V'(s)\Big), \tag{2.7}$$

possibly shrinking I_0 to guarantee that $V(s)+(s/2)V'(s) \neq 0$, so φ is smooth. Hence κ and N are smooth on I_0, and

$$T'(s) = \kappa(s)N(s), \quad N(s) \perp T(s), \quad |N(s)| = 1. \tag{2.8}$$

Note that, by (2.6),

$$\kappa(s) \text{ is negative for } s < s_0,\ 0 \text{ for } s = s_0, \text{ positive for } s > s_0. \tag{2.9}$$

As an alternative, we could change the signs of κ and N, still obtaining (2.8), but with the signs in (2.9) reversed.

Having smooth T, N, and κ satisfying (2.8), we set

$$B(s) = T(s) \times N(s), \tag{2.10}$$

as in (1.13), and via calculations parallel to (1.14)–(1.16), obtain the Frenet-Serret equations, which again can be presented as in (1.17) or as in (1.18)–(1.21).

Generally, if $\gamma : I \to \mathbb{R}^3$ is smooth and (1.1) holds, and if $T'(s)$ is nonvanishing except at points $s_1, \dots, s_\ell \in I = (a, b)$, where T' vanishes simply, one can make a preliminary definition of the fields $N(s)$ and $B(s)$, and the functions κ and τ, on each interval

$$(a, s_1), (s_1, s_2), \dots, (s_{\ell-1}, s_\ell), (s_\ell, b), \tag{2.11}$$

using the method of (1.11)–(1.15). In such a definition, $\kappa > 0$ on each such interval, tending to 0 at each endpoint s_j. The vector fields $N(s)$ and $B(s)$ will change sign across each such endpoint. One can change the signs of these vector fields, on alternating intervals, so that they fit together to form smooth vector fields on $I = (a, b)$. Doing this changes the sign of κ across each endpoint s_j. On the last interval (s_ℓ, b), the adjusted κ will be negative if ℓ is odd and positive if ℓ is even (assuming κ starts out positive on (a, s_1)). On the other hand, τ is unaffected.

As a variant, suppose γ is periodic, of period A (so A is the length of the resulting closed curve in $\mathbb{R}^3$). We can write

$$\gamma : S \longrightarrow \mathbb{R}^3, \quad S = \mathbb{R}/A\mathbb{Z}, \tag{2.12}$$

and similarly $T : S \to \mathbb{R}^3$. Assume T' vanishes precisely at $s_1, \dots, s_\ell \in S$, to first order at each such point. We can define

$$\Phi : [0, A] \times (-\varepsilon, \varepsilon) \to \mathbb{R}^3, \quad \Phi(s, t) = \gamma(s) + tN(s). \tag{2.13}$$

Now

$$\ell \text{ even } \Rightarrow N(A) = N(0), \quad \ell \text{ odd } \Rightarrow N(A) = -N(0). \tag{2.14}$$

We see that if ε is sufficiently small, then if ℓ is even the image of $[0, A] \times (-\varepsilon, \varepsilon)$ is diffeomorphic to an annulus, while if ℓ is odd, the image is diffeomorphic to a Mobius strip.

3 Extension to curvature vectors vanishing to finite order

Generalizing (2.3), take $k \geq 2$, suppose $\gamma : I \to \mathbb{R}^3$ is smooth, with tangent $T(s) = \gamma'(s)$ satisfying (1.1), that $T'(s_0) = 0$, and (taking $s_0 = 0$),

$$T(s) = T_0 + \frac{s^k}{k} V(s), \quad V(0) = V_0 \neq 0. \tag{3.1}$$

where $V : I_0 \to \mathbb{R}^3$ is smooth and $|T_0| = 1$. As in (2.4), we have $T_0 \cdot V_0 = 0$. Parallel to (2.5), we have

$$T'(s) = s^{k-1} V(s) + \frac{s^k}{k} V'(s). \tag{3.2}$$

Hence we define κ and N on I_0 by

$$\kappa(s) = s^{k-1} \Big| V(s) + \frac{s}{k} V'(s) \Big| = s^{k-1} \varphi(s), \tag{3.3}$$

and

$$N(s) = \frac{1}{\kappa(s)} T'(s) = \frac{1}{\varphi(s)} \Big(V(s) + \frac{s}{k} V'(s) \Big), \tag{3.4}$$

possibly shrinking I_0 to guarantee that $V(s) + (s/k)V'(s) \neq 0$, so φ is smooth. Hence κ and N are smooth on I_0, and

$$T'(s) = \kappa(s) N(s), \quad N(s) \perp T(s), \quad |N(s)| = 1. \tag{3.5}$$

Note that

$$\kappa(s) \text{ changes sign as } s \text{ crosses } s_0 \Leftrightarrow k \text{ is even}. \tag{3.6}$$

Again we define smooth $B(s)$ by

$$B(s) = T(s) \times N(s), \tag{3.7}$$

obtaining an orthonormal frame for $s \in I_0$, on which the Frenet-Serret equations hold.

4 Counterexamples with curvature vector vanishing to infinite order

Suppose $\gamma : I \to \mathbb{R}^3$ is smooth, with tangent $T(s) = \gamma'(s)$ satisfying (1.1), so we have a smooth map

$$T : I \longrightarrow S^2. \tag{4.1}$$

It $T'(s_0)$ vanishes to infinite order at $s_0 \in I$, an analysis such as in (3.1) does not apply. Here we give examples where one cannot define the normal field N to be continuous at s_0. We deal directly with with T, as in (4.1). One can then use (1.10) to specify γ.

To proceed, pick $p \in S^2$, and consider two smooth, unit-speed curves $\widetilde{T}_1, \widetilde{T}_2 : [0,1] \to S^2$, with $\widetilde{T}_j(0) = p$, such that the angle θ between these curves at p satisfies $0 < \theta < \pi$. One can reparametrize, obtaining curves T_j, so that $T_1'(s)$ and $T_2'(s)$ vanish to infinite order at $s = 0$, but are otherwise non-vanishing. Then we can set

$$\begin{aligned} T(s) = T_1(s), &\qquad \text{for } \ 0 \le s \le 1, \\ T_2(-s), &\quad \text{for } -1 \le s \le 0, \end{aligned} \tag{4.2}$$

obtaining a smooth map as in (4.1), with $I = [-1, 1]$. Now, on each segment T_j, we can write

$$T_j'(s) = \kappa_j(s) N_j(s), \quad |N_j(s)| = 1, \ \kappa_j(0) = 0, \ \kappa_j(s) > 0 \text{ for } s > 0, \tag{4.3}$$

where each N_j is continuous on $[0, 1]$, and the angle between $N_1(0)$ and $-N_2(0)$ is θ. Hence, for $s \neq 0$,

$$T'(s) = \kappa(s) N(s), \tag{4.4}$$

with

$$\begin{aligned} N(s) = \ &\pm N_1(s), \qquad \text{for } \ 0 < s < 1, \\ &\pm N_2(-s), \quad \text{for } -1 \le s < 0. \end{aligned} \tag{4.5}$$

However, no choice of signs can allow for N to be defined at $s = 0$ so as to be continuous.

Reference

[1] M. Taylor, Introduction to Differential Equations (2nd ed.), Undergrad Texts #52, Amer. Math. Soc., Providence RI, 2022.