Microlocal analysis on contact manifolds: Eigenfunction concentration, Weyl formulas, and Noncommutative phase space

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Notes available on my website, under Downloadable Lecture Notes

8. Seminar talks and AMS talks

See also

1. Pseudodifferential operators, Fourier integral operators, and microlocal analysis

Heat asymptotics and microlocal Weyl asymptotics on a compact Riemannian manifold M, with Laplace operator Δ.

Orthonormal basis φ_k, eigenvalues $-\lambda_k$.

$$\operatorname{Tr} e^{t\Delta} = \sum_{l \ge 0} e^{-\lambda_k t}, \quad \operatorname{Tr} A e^{t\Delta} = \sum_{k \ge 0} e^{-\lambda_k t} (A\varphi_k, \varphi_k)_{L^2}.$$

$$A \in \mathrm{OPS}^0(M).$$

Heat asymptotics

$$\operatorname{Tr} e^{t\Delta} \sim C(M) t^{-m/2}, \quad m = \dim M.$$

$$\operatorname{Tr} A e^{t\Delta} \sim C(M) \Bigl(\int_{S^*M} \sigma_A(z,\zeta)\, dS(z,\zeta) \Bigr) t^{-m/2}.$$

Heat parametrix

$$e^{t\Delta} f(z) \sim \int_{T^*M} h(t,z,\zeta) e^{-tG_2(z,\zeta)} e^{i(z-w)\cdot\zeta} f(w)\, dw\, d\zeta.$$

$$\operatorname{Tr} e^{t\Delta} \sim \int_{T^*M} e^{-tG_2(z,\zeta)} dV(z,\zeta).$$

Similar formulas for $Ae^{t\Delta}$, given action of A:

$$Af(z) = (2\pi)^{-n} \int a(z,\zeta) e^{i(z-w)\cdot\zeta} f(w)\, dw\, d\zeta.$$

Microlocal Weyl formula, via Tauberian theorem:

$$\lim_{n\to\infty} \frac{1}{N} \sum_{k=1}^{N} (A\varphi_k, \varphi_k)_{L^2} = \int_{S^*M} \sigma_A(z,\zeta)\, dS(z,\zeta).$$

Corollary: local Weyl formula: take special case $Af(z) = a(z)f(z)$.

$$\lim_{n\to\infty} \frac{1}{N} \sum_{k=1}^{N} \int_M a(z)|\varphi_k(z)|^2\, dV(z) = \frac{1}{V(M)} \int_M a(z)\, dV(z).$$

On to hypoelliptic operators.

L second order differential operator on M, formally negative and self adjoint. Assume hypoelliptic with loss of < 2 derivatives.

Hence, for some $\sigma > 0$,

$$L^{-1} : L^2(M) \longrightarrow H^\sigma(M).$$

Thus L^{-1} compact and self adjoint on $L^2(M)$, so have ON basis $\{\varphi_k\}$ of eigenfunctions:

$$L\varphi_k = -\lambda_k \varphi_k, \quad 0 \le \lambda_k \nearrow +\infty.$$

Theorem. Let L_2 denote the principal symbol of L. If

$$\int_{S^*M} |L_2(z,\zeta)|^{-m/2}\, dS(z,\zeta) = \infty,$$

then, except for a sparse subsequence, $\{\varphi_k\}$ concentrates microlocally on the characteristic set of $\Sigma \subset T^*M \setminus 0$, given by

$$\Sigma = \{(z,\zeta) \in T^*M \setminus 0 : L_2(z,\zeta) = 0\}.$$

Ingredients in proof: if A_0 vanishes on conic nbd of Σ,

$$\operatorname{Tr} Ae^{tL} \sim C(A_0)t^{-m/2}, \quad \text{as } t \searrow 0,$$

where $A_0 =$ total symbol of A, but

$$t^{m/2} \operatorname{Tr} e^{tL} \longrightarrow +\infty, \quad \text{as } t \searrow 0.$$

Proof of last result uses

$$\operatorname{Tr} e^{tL} \geq \operatorname{Tr} e^{t(L+\varepsilon\Delta)},$$

plus heat asymptotics in elliptic case.

Taking $A \mapsto A^*A$, with A as above, yields

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} \|A\varphi_k\|_{L^2}^2 = 0,$$

hence the theorem.

Contact manifold, with contact line bundle Σ

M compact manifold of dimension $m = 2n + 1$, $\Sigma \subset T^*M$ line bundle, κ section of Σ (one-form). Contact condition:

$$\kappa \wedge (d\kappa)^n \text{ nowhere vanishing.}$$

Complementary subbundle HM of TM.

$$H_z M = \{X \in T_z M : X(z) \perp \kappa(z)\}.$$

X_j, X_k smooth sections of $HM \Rightarrow$

$$d\kappa(X_j, X_k) = -\kappa([X_j, X_k]).$$

Contact condition $\Rightarrow$

$$d\kappa(z) \text{ nondegenerate bilinear form on } H_z M,$$

so we have maximal non-involutivity of HM.

Examples of contact manifolds.

$M \subset \mathbb{R}^{2\mu} \approx \mathbb{C}^{\mu}$, codim 1. N outward unit normal, J complex structure. Define

$$\kappa(X) = \langle X, JN \rangle, \quad H_z M = \{X \in T_z M : X \perp JN(z)\}.$$

As before, X, Y smooth sections of $HM \Rightarrow$

$$-d\kappa(X, Y) = \kappa([X, Y]).$$

Contact condition:

$$d\kappa(z) \text{ nondegenerate bilinear form on } H_z M, \ \forall\, z \in M.$$

Levi form (symmetric)

$$L(X, Y) = -d\kappa(X, JY), \quad X, Y \in H_z M.$$

Geometric formula (see [T3], Chapter 12, or [T6], §3):

$$L(X, X) = \mathrm{II}(X, X) + \mathrm{II}(JX, JX).$$

II = second fundamental form of $M \subset \mathbb{R}^{2\mu}$. $\mathrm{II}(X, Y) = \langle A_N X, Y \rangle$.

$A_N X = -D_X N, \quad N : M \to S^{2\mu-1}$, Gauss map, A_N Weingarten map.

Definition. If $M = \partial\Omega$ and L is positive definite, we say Ω is a strongly pseudoconvex domain.

Key example: Heisenberg group.

$$\mathbb{H}^n = \{(t, q, p) : t \in \mathbb{R},\ q, p \in \mathbb{R}^n\}.$$

Group law

$$\begin{aligned} &(t_1, q_1, p_1) \cdot (t_2, q_2, p_2) \\ &= (t_1 + t_2 + \tfrac{1}{2}\sigma((q_1, p_1), (q_2, p_2)), q_1 + q_2, p_1 + p_2). \end{aligned}$$

Symplectic form on $\mathbb{R}^{2n}$:

$$\sigma((q_1, p_1), (q_2, p_2)) = p_1 \cdot q_2 - q_1 \cdot p_2.$$

Let algebra $\mathfrak{h}^n$ of right-invariant vector fields:

$$T = \frac{\partial}{\partial t},\ L_j = \frac{\partial}{\partial q_j} - \frac{p_j}{2}\frac{\partial}{\partial t},\ M_j = \frac{\partial}{\partial p_j} + \frac{q_j}{2}\frac{\partial}{\partial t},$$

with $1 \le j \le n$. Commutator relations

$$[L_j, M_j] = -[M_j, L_j] = T,$$

other commutators zero.

Contact structure of $\mathbb{H}^n$:

$$H_z\mathbb{H}^n = \mathrm{Span}\{X_j : 1 \le j \le 2n\}, \quad X_j = L_j,\ X_{n+j} = M_j.$$
$$\kappa = dt + \tfrac{1}{2}(q \cdot dp - p \cdot dq).$$

Automorphisms of $\mathbb{H}^n$:

$$\delta_{\pm\lambda}(t, q, p) = (\pm\lambda t, \pm\lambda^{1/2} q, \lambda^{1/2} p), \quad \lambda > 0,$$
$$\alpha_g(t, q, p) = (t, g(q, p)), \quad g \in \mathrm{Sp}(n, \mathbb{R}).$$

Representations of $\mathbb{H}^n$ on $L^2(\mathbb{R}^n)$:

$$\pi_{\pm\lambda}(t, q, p) = \pi_1(\delta_{\pm\lambda}(t, q, p)),$$
$$\pi_1(t, q, p) = e^{i(t + q\cdot X + p\cdot D)}.$$

Here $q \cdot Xu(x) = \sum q_j x_j u(x)$, $p \cdot Du(x) = i^{-1} \sum p_j \partial u/\partial x_j$.
Also have scalar reps, $\pi_{(y,\eta)}(t, q, p) = e^{i(y\cdot q + \eta\cdot p)}$.

Sublaplacians on a contact manifold M, dimension $m = 2n + 1$.

Second order differential operator L, principal symbol

$$L_2 \leq 0 \text{ on } T^*M \setminus 0, \text{ vanishing precisely on } \Sigma.$$

Proposition. One-to-one correspondence between such principal symbols and smooth, positive-definite inner products on the vector bundle HM.

Over open $\mathcal{O} \subset M$, with frame field $\{X_j\}$ for HM defined over $\mathcal{O}$, get operator with principal symbol L_2:

$$L^b = \sum_{j,k} a_{jk}(z) X_j X_k,$$

plus first-order operator. Here $(a_{jk}(z))$ positive definite.

Formally self-adjoint modification (assume M is given a volume form):

$$\begin{aligned} L^{\#} &= -\sum_{j,k} X_j^* a_{jk}(z) X_k, \quad -X_j^* u = X_j u + \tilde{b}_j u \\ &= L^b + \sum_k b_k X_k, \quad b_k = \sum_j \tilde{b}_j a_{jk}. \end{aligned}$$

General case:

$$L = L^{\#} + iY + c,$$

Y real vector field, $iY + c$ formally self-adjoint.
Real coefficients $\Rightarrow Y \equiv 0$.

Darboux theorem for contact manifolds.

M, M' m-dimensional contact manifolds, containing p, p'. There are neighborhoods $\mathcal{O}, \mathcal{O}'$, and a contact diffeomorphism

$$F : \mathcal{O} \longrightarrow \mathcal{O}'.$$

We have

$$DF(z) : H_z\mathcal{O} \xrightarrow{\approx} H_{z'}\mathcal{O}',$$

for $z \in \mathcal{O},\ z' = F(z)$.

Apply this with $M' = \mathbb{H}^n,\ m = 2n+1$. Get

$$L = \sum_{j,k=1}^{2n} a_{jk}(z) X_j X_k + i\alpha(z) T + B,$$

with

$$X_j = L_j, \quad X_{n+j} = M_j, \quad 1 \le j \le n,$$
$$B = \sum_j b_j(z) X_j + c(z).$$

Heisenberg calculus

First treat right-invariant operators,

$$L = \sum_{j,k} a_{jk} X_j X_k + i\alpha T, \quad a_{jk}, \alpha \in \mathbb{R}.$$

Consider convolution operators on $\mathbb{H}^n$, $Ku = k * u$.

$$\begin{aligned}\pi_{\pm\lambda}(k) &= \int k(t,q,p)\pi_{\pm\lambda}(t,q,p)\,dt\,dq\,dp \\ &= \hat{k}(\pm\lambda, \pm\lambda^{1/2}X, \lambda^{1/2}D),\end{aligned}$$

$\hat{k}$ Euclidean Fourier transform of k. Use Weyl calculus:

$$a(X,D) = \int \hat{a}(q,p) e^{i(q\cdot X + p\cdot D)}\,dq\,dp,$$

i.e., for $f \in \mathcal{S}(\mathbb{R}^n)$,

$$a(X,D)f(x) = (2\pi)^{-n} \iint e^{i(x-y)\cdot\xi} a(\tfrac{1}{2}(x+y), \xi) f(y)\,dy\,d\xi.$$

Also write $\sigma_K(\pm\lambda)(X, D) = \pi_{\pm\lambda}(k)$, so key identity

$$\pi_\lambda(k * u) = \pi_\lambda(k)\pi_\lambda(u)$$

implies

$$\pi_{\pm\lambda}(Ku) = \sigma_K(\pm\lambda)(X, D)\pi_{\pm\lambda}(u).$$

Key cases:

$$\sigma_T(\pm\lambda)(X, D) = \pm i\lambda I, \quad \sigma_{L_j}(\pm\lambda)(X, D) = \pm i\lambda^{1/2} x_j,$$

$$\sigma_{M_j}(\pm\lambda)(X, D) = \lambda^{1/2}\frac{\partial}{\partial x_j}.$$

Definition. For $m \in \mathbb{R}$, Ψ_0^m consists of functions $\hat{k}(\tau, y, \eta)$, smooth on $\mathbb{H}^n \setminus 0$, such that for $\tau > 0$,

$$\hat{k}(\pm\tau, y, \eta) = \tau^{m/2}\hat{k}(\pm 1, \tau^{-1/2}y, \tau^{-1/2}\eta).$$

We say

$$Kw = k * w, \ \hat{k} \in \Psi_0^m \Rightarrow K \in \mathrm{OP}\,\Psi_0^m.$$

In such a case,

$$\sigma_K(\pm\lambda)(x, \xi) = \lambda^{m/2}\sigma_K(\pm 1)(x, \xi).$$

Key case:

$$L = \sum_{jk} a_{jk} X_j X_k + i\alpha T \Rightarrow L \in \mathrm{OP}\,\Psi_0^2,$$

and

$$\sigma_L(\pm\lambda)(X, D) = -\lambda H_{\pm}, \quad H_{\pm} = Q(\pm X, D) \pm \alpha I,$$
$$Q(x, \xi) = \sum_{j,k} a_{jk} \chi_j \chi_k, \quad \chi_j = x_j,\ \chi_{n+j} = \xi_j.$$

Special case:

$$L = \sum_{j=1}^{2n} X_j^2 \Rightarrow Q(x, \xi) = |x|^2 + |\xi|^2$$
$$\Rightarrow Q(\pm X, D) = H_0 = -\Delta + |x|^2,$$

n-dimensional harmonic oscillator Hamiltonian.

Classical spectral calculation:

$$n = 1 \Longrightarrow \operatorname{Spec} H_0 = \{1, 3, 5, \dots\}.$$

Lemma. If $Q(x, \xi)$ is a positive quadratic form on $\mathbb{R}^{2n}$, $\exists\, g \in \mathrm{Sp}(n, \mathbb{R})$ such that

$$Q(g(x, \xi)) = \sum_{j=1}^{n} \mu_j (x_j^2 + \xi_j^2), \quad \mu_j > 0.$$

Metaplectic covariance. Given $g \in \mathrm{Sp}(n, \mathbb{R})$,

$$\begin{aligned} Q_g(x, \xi) = Q(g(x, \xi)) &\Rightarrow Q_g(X, D) \text{ unitarily equiv to } Q(X, D) \\ &\Rightarrow \operatorname{Spec} Q(\pm X, D) = \operatorname{Spec} Q_g(X, D). \end{aligned}$$

Corollary.

$$\operatorname{Spec} Q(\pm X, D) = \Big\{ \sum_j \mu_j (2k_j + 1) : k_j \in \mathbb{Z}^+ \Big\}.$$

Proposition. Provided $\pm\alpha$ satisfies the condition for invertibility of $H_\pm$, we have

$$E \in \mathrm{OP}\,\Psi_0^{-2} \ \text{ such that } \ \sigma_E(\pm\lambda)(X, D) = \lambda^{-1} H_\pm^{-1},$$

and E is a two-sided parametrix for $L^b = \sum a_{jk} X_j X_k + i\alpha T$.

Note. We have

$$\begin{aligned} \mathrm{OP}\,\Psi_0^m \subset \ & \mathrm{OPS}^m_{1/2,1/2}, \text{ if } m \geq 0, \\ & \mathrm{OPS}^{m/2}_{1/2,1/2}, \text{ if } m < 0. \end{aligned}$$

Corollary. The existence of the parametrix E implies that L^b is hypoelliptic, with loss of one derivative. This holds particularly if

$$|\alpha| < \sum \mu_j, \ \text{ implying } H_\pm \text{ positive definite.}$$

Lower order terms

$$L = \sum a_{jk} X_j X_k + i\alpha T + \sum b_j X_j + c = L^b + B_1 + B_0.$$

Take $E \in \mathrm{OP}\,\Psi_0^{-2}$, parametrix for L^b. Then

$$EL = I + E(B_1 + B_0), \quad LE = I + (B_1 + B_0)E,$$

and $X_j \in \mathrm{OP}\,\Psi_0^1, c \in \mathbb{C} \Rightarrow$

$$EB_1, B_1 E \in \mathrm{OP}\,\Psi_0^{-1}, \quad EB_0, B_0 E \in \mathrm{OP}\,\Psi_0^{-2}.$$

Now

$K_j \in \mathrm{OP}\,\Psi_0^{m_j} \Rightarrow K_1 K_2 \in \mathrm{OP}\,\Psi_0^{m_1+m_2}$, and

$$\sigma_{K_1 K_2}(\pm\lambda)(X, D) = \sigma_{K_1}(\pm\lambda)(X, D)\sigma_{K_2}(\pm\lambda)(X, D).$$

Definition. $K \in \mathrm{OP}\,\Psi^m$ provided

$$K \sim \sum_{j\ge 0} K_j, \quad K_j \in \mathrm{OP}\,\Psi_0^{m-j},$$

in the sense that $K - \sum_{j\le N} K_j$ is arbitrarily smoothing, for N large.

Proposition. Given that L^b has a parametrix $E^b \in \mathrm{OP}\,\Psi_0^{-2}$,

$$L \text{ has a parametrix } E \in \mathrm{OP}\,\Psi^{-2}.$$

Hence L is hypoelliptic, with loss of one derivative.

Other operator classes. Given $K \in \mathrm{OP}\,\Psi_0^m,\ k \in \mathbb{Z}^+$,

$$K \in \mathrm{OP}\,\Psi_0^{m,k} \Leftrightarrow \sigma_K(\pm 1)(x,\xi) = a_\pm(x,\xi),\ \text{with}$$
$$a_\pm(x,\xi) \sim \sum_{j\ge k} (-1)^j \varphi_j(\pm x,\xi),\ \text{as } |x|^2+|\xi|^2 \to \infty,$$
$$\varphi_j \in C^\infty(\mathbb{R}^{2n}\setminus 0),\ \text{homogenepus of degree}\ \ m-2j.$$

We write $a_\pm(X,D) \in \mathrm{OP}\,\mathcal{H}^{m,k}(\mathbb{R}^n)$.

Note: $T \in \mathrm{OP}\,\Psi_0^{2,1}$.

$$K_j \in \mathrm{OP}\,\Psi_0^{m_j} \Rightarrow [K_1,K_2] \in \mathrm{OP}\,\Psi_0^{m_1+m_2,1}.$$
$$K \in \mathrm{OP}\,\Psi_0^{m,k} \Rightarrow K \in \mathrm{OP}S^{m-k},\ \text{microlocally away from } \Sigma.$$

Also set

$$\mathrm{OP}\,\Psi_0^{m,\infty} = \bigcap_k \mathrm{OP}\,\Psi_0^{m,k}.$$

Variable coefficients.
$K(z)$ smooth family of elements of $\mathrm{OP}\,\Psi^m$. Set

$$Ku(z) = K(z)u(z), \quad \text{yielding } K \in \mathrm{OP}\,\widetilde{\Psi}^m.$$

Similarly define $\mathrm{OP}\,\widetilde{\Psi}^{m,k}$. Set $K_0(z) =$ part in $\mathrm{OP}\,\Psi^m_0$, and

$$\sigma_K(z, \pm\lambda)(X, D) = \sigma_{K_0(z)}(\pm\lambda)(X, D).$$

Theorem. Operator and symbol calculus for $\mathrm{OP}\,\widetilde{\Psi}^m$ work. In particular,

$$K_j \in \mathrm{OP}\,\widetilde{\Psi}^{m_j} \Rightarrow A = K_1K_2 \in \mathrm{OP}\,\widetilde{\Psi}^{m_1+m_2}, \text{and}$$
$$A(z) - K_1(z)K_2(z) \in \mathrm{OP}\,\Psi^{m_1+m_2-1}, \ \forall\, z, \ \text{hence}$$
$$\sigma_{K_1K_2}(z, \pm\lambda)(X, D) = \sigma_{K_1}(z, \pm\lambda)(X, D)\sigma_{K_2}(z, \pm\lambda)(X, D).$$

Variable coefficient hypoellipticity

$$\begin{aligned} L &= \sum_{j,k} a_{jk}(z) X_j X_k + i\alpha(z) T + \sum_j b_j(z) X_j + c(z) \\ &= L^b + B_1 + B_2. \end{aligned}$$

$$L \in \mathrm{OP}\,\widetilde{\Psi}^2, \quad B_1 \in \mathrm{OP}\,\widetilde{\Psi}^1, \quad B_2 \in \mathrm{OP}\,\widetilde{\Psi}^0.$$

Symbol computation,

$$\sigma_L(z, \pm\lambda)(X, D) = -\lambda H_\pm(z), \quad H_\pm(z) = Q_z(\pm X, D) \pm \alpha(z).$$

Theorem. Given $(a_{jk}(z)) \in M(2n, \mathbb{R})$ positive definite,
$H_\pm(z) \in \mathrm{OP}\,\mathcal{H}^2(\mathbb{R}^n)$ is elliptic.
Then L has a parametrix $E \in \mathrm{OP}\,\widetilde{\Psi}^{-2}$ provided $H_\pm(z)$ is invertible, for each z.
$H_\pm(z)$ is positive definite provided $|\alpha(z)| < \sum \mu_j(z)$.

Sublaplacians on a compact contact manifold, M, of dimension $m = 2n + 1$.

L second order differential operator, formally self-adjoint and negative semidefinite, principal symbol $L_2 \le 0$, vanishing precisely on contact line bundle Σ.

In Heisenberg coordinates, L has form described in last frame.

Assume $|\alpha(z)| < \sum \mu_j(z)$, so $H_\pm(z)$ is positive-definite.

Then $L^{-1} : L^2(M) \to H^1(M)$, so $L^2(M)$ has an orthonormal basis φ_k of eigenfunctions:

$$L\varphi_k = -\lambda_k \varphi_k, \quad \lambda_k \nearrow \infty.$$

Heat trace asymptotics (cf. [MS]):

$$\operatorname{Tr} e^{tL} \sim C_1 t^{-n-1}.$$

Yields eigenvalue asymptotics,

$$\#\{k : \lambda_k \le \lambda\} \sim C_2 \lambda^{n+1}.$$

Microlocal Weyl formula, in standard ψDO calculus.
Microlocal heat trace asymptotics:

$$A \in \mathrm{OPS}^0(M) \Rightarrow \mathrm{Tr}\, Ae^{tL} \sim C_1 I(\sigma_A) t^{-n-1},$$
$$I(\sigma_A) = \int_M \sum_{\pm} \sigma_A(\pm\kappa(z)) K_{\pm}(z)\, dV_0(z).$$

Weyl law (cf. [CHT]):

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} (A\varphi_k, \varphi_k)_{L^2} = I(\sigma_A).$$

Apply with $A \mapsto A^*A$ to get

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} \|A\varphi_k\|_{L^2}^2 = I(|\sigma_A|^2).$$

Implies a concentration of eigenfunctions microlocally on Σ.

$$\sigma_A\big|_{\Sigma} = 0 \Rightarrow I(\sigma_A) = I(|\sigma_A|^2) = 0.$$

Refined microlocal Weyl formula, in Heisenberg calculus.

Bring in symbol

$$\tilde{\sigma}_A : \Sigma \setminus 0 \longrightarrow \mathrm{OP}\,\mathcal{H}^0(\mathbb{R}^n), \quad A \in \mathrm{OP}\,\widetilde{\Psi}^0(M).$$

Theorem. Cf. [T4]. Take

$$A \in \mathrm{OP}\,\widetilde{\Psi}^0(M).$$

Then

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} (A\varphi_k, \varphi_k)_{L^2}$$
$$= C(M) \int_M \sum_{\pm} \mathrm{Tr}\,\tilde{\sigma}_A(\pm\kappa(z)) H_\pm(z)^{-n-1}\, dV_0(z).$$

Proof uses asymptotics for $\mathrm{Tr}\, Ae^{tL}$, using parametrix for e^{tL} via a different approach from [MS], using Heisenberg calculus, produced independently by [T1] and [BGS], [BG].
Related theory in [P1], [P2] can also be applied.

Further refined microlocal Weyl formula.
Use one-parameter families

$$A_\delta \in \mathrm{OP}\,\widetilde{\Psi}^{0,\infty}.$$

Example. $\tilde{\sigma}_{A_\delta}(\pm\kappa(z))(X, D) = \psi(\delta H_\pm(z)),$

$$\psi \in \mathcal{S}(\mathbb{R}), \quad \psi(\lambda) = 1 \text{ for } |\lambda| \le 1.$$

Proposition 1. Given such A_δ,

$$\lim_{N\to\infty} \frac{1}{N}\sum_{k=1}^{N} (A_\delta \varphi_k, \varphi_k)_{L^2} = 1 + R(\delta), \quad R(\delta) \sim \sum_{j\ge 1} c_j \delta^j.$$

Proof uses

$$|\operatorname{Tr} \psi_b(\delta H_\pm(z)) H_\pm(z)^{-n-1}| \le c\delta, \quad \psi_b(\lambda) = 1 - \psi(\lambda).$$

Refined microlocal concentration result.

Proposition 2. Given such A_δ as in Proposition 1,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} \|(I - A_\delta)\varphi_k\|_{L^2}^2 \le C\delta.$$

Proof uses Prop.1 and takes $I - A_\delta \mapsto (I - A_\delta)^*(I - A_\delta)$.

Other examples: $A_\varepsilon \in \mathrm{OP}\,\widetilde{\Psi}^{0,\infty}(M)$.

$$\tilde{\sigma}_{A_\varepsilon}(\pm\kappa(z))(X, D) = a_z^\pm(\varepsilon X, \varepsilon D), \quad a_z^\pm \in \mathcal{S}(\mathbb{R}^{2n}).$$

Similar results. Bring in semiclassical analysis.

More general case of $A_\varepsilon \in \mathrm{OP}\,\widetilde{\Psi}^{0,\infty}(M)$

$$\tilde{\sigma}_{A_\varepsilon}(\pm\kappa(z))(X, D) = a_z^\pm(\varepsilon^2, \varepsilon X, \varepsilon D),$$

$a_z^\pm(\varepsilon^2, \cdot, \cdot)$ smooth in ε^2, values in $\mathcal{S}(\mathbb{R}^{2n})$.
This class also contains the class A_δ, with $\delta = \varepsilon^2$.

Proposition 3. Given $\psi \in \mathcal{S}(\mathbb{R})$, $H = Q(X, D) + \alpha I$ positive elliptic, $Q(\zeta)$ positive quadratic form,

$$\psi(\delta H) = B_H(\varepsilon^2, \varepsilon X, \varepsilon D),$$

with $B_H(\varepsilon^2, \zeta)$ smooth in ε^2 with values in $\mathcal{S}(\mathbb{R}^{2n})$, $\delta = \varepsilon^2$, $\zeta = (x, \xi)$. Furthermore,

$$B_H(\varepsilon^2, \zeta) = \sum_{j,k\ge 0} \sum_{|\beta|=2k} \varepsilon^{2(j+k)} b_{jk\beta} \zeta^\beta \psi^{(j+2k)}(Q(\zeta)).$$

Leading term is $\psi(Q(\zeta))$.

Proof of Prop. 3 uses

$$\psi(\delta H) = \int \hat{\psi}_\delta(t) e^{itH}\, dt, \quad \psi_\delta(\lambda) = \psi(\delta\lambda),$$

together with Fourier integral representation of the Schrödinger semigroup e^{itH}, obtained via analytic continuation from formula for "heat" semigroup

$$\begin{aligned} e^{-\tau Q(X,D)} &= h^Q_\tau(X,D), \\ h^Q_\tau(x,\xi) &= (\det\cosh \tau A_Q)^{-1/2} e^{-\tau Q(\vartheta(\tau A_Q)\zeta,\zeta)}, \end{aligned}$$

with $\zeta = (x,\xi)$, $A_Q = (-F_Q^2)^{1/2}$, $\sigma(u, F_Q v) = Q(u,v)$,

$$\vartheta(t) = t^{-1} \tanh t.$$

Variant of Mehler formula.

Contact with Weyl formula involving $P \in \mathrm{OPS}^0(M)$.

Lemma 4. If $A \in \mathrm{OP}\,\widetilde{\Psi}^{0,\infty}(M)$, $P \in \mathrm{OPS}^0(M)$, then $PA \in \mathrm{OP}\,\widetilde{\Psi}^{0,\infty}$, and

$$\tilde{\sigma}_{PA}(\pm\kappa(z))(X, D) = \sigma_P(\pm\kappa(z))\tilde{\sigma}_A(\pm\kappa(z))(X, D),$$

where $\sigma_P(\pm\kappa(z))$ = classical principal symbol of P, restricted to $\Sigma \setminus 0$.

Corollary 5. Given such P, A_δ,

$$\begin{aligned}\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} &(PA_\delta\varphi_k, \varphi_k)_{L^2} \\ &= C(M) \int_M \sum_{\pm} \sigma_P(\pm\kappa(z))\, \mathrm{Tr}\, \psi(\delta H_\pm(z)) H_\pm(z)^{-n-1}\, dV_0(z).\end{aligned}$$

Taking $\delta \to 0$ and using Prop. 2 yields Weyl formula of [CHT], with $K_\pm(z) = \mathrm{Tr}\, H_\pm(z)^{-n-1}$.

Remark. If L has real coefficients, then $\alpha \equiv 0$ and $K_+(z) = K_-(z)$.

Appendix.

Heisenberg-calculus-microlocalized heat trace asymptotics. Model case.

$$\sigma_L(\pm\lambda)(X,D) = -H_\pm, \quad H_\pm = Q(\pm X, D) \pm \alpha I.$$

Take $A \in \mathrm{OP}\,\Psi^0_0$.

$$\begin{aligned} Ae^{sL}\delta_0(t,q,p) &= k_{A,s}(t,q,p), \\ \hat{k}_{A,s}(\pm\tau, y, \eta) &= \sigma_{Ae^{sL}}(\pm\tau)(\pm\tau^{-1/2}y, \tau^{-1/2}\eta), \\ \sigma_{Ae^{sL}}(\pm\tau)(X,D) &= \sigma_A(\pm1)(X,D)e^{-\tau s H_\pm}. \end{aligned}$$

To write local trace of Ae^{sL}, compute

$$k_{A,s}(0,0,0) = \iiint \hat{k}_{A,s}(\tau,y,\eta)\,dy\,d\eta\,d\tau.$$

To start,

$$\begin{aligned}\iint \hat{k}_{A,s}(\pm\tau, y, \eta)\, dy\, d\eta &= \iint \sigma_{Ae^{sL}}(\pm\tau)(\pm\tau^{-1/2}y, \tau^{-1/2}\eta)\, dy\, d\eta \\ &= \tau^n \iint \sigma_{Ae^{sL}}(\pm\tau)(x,\xi)\, dx\, d\xi \\ &= \tau^n \operatorname{Tr} \sigma_{Ae^{sL}}(\pm\tau)(X, D) \\ &= \tau^n \operatorname{Tr} \sigma_A(\pm 1)(X, D) e^{-\tau s H_\pm}.\end{aligned}$$

Hence

$$\begin{aligned}k_{A,s}(0,0,0) &= \sum_\pm \tau^n \operatorname{Tr} \sigma_A(\pm 1)(X, D) e^{-\tau s H_\pm}\, d\tau \\ &= s^{-n-1} \sum_\pm \int_0^\infty t^n \operatorname{Tr} \sigma_A(\pm 1)(X, D) e^{-tH_\pm}\, dt \\ &= C_n s^{-n-1} \sum_\pm \operatorname{Tr} \sigma_A(\pm 1)(X, D) H_\pm^{-n-1}.\end{aligned}$$

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