Note on Evaluating Square Roots

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People have been interested in extracting square roots for a long time. As noted in a recent Notices article [K], a translation of a Babylonian clay tablet from around 1800 BC contains the formula

$$\sqrt{2} \approx s = 1.41421297, \tag{1}$$

which agrees within $\approx 6 \times 10^{-7}$ with the actual value,

$$\sqrt{2} \approx 1.414213562373095 \cdots . \tag{2}$$

The tablet does not discuss the method used to obtain (1), and the article mentions that some think the method used might be related to what is sometimes called the Babylonian method. For this method to approximate $\sqrt{a}$, given $a > 0$, one starts with an initial guess $x_0 > 0$ and recursively takes

$$x_{k+1} = \frac{1}{2}\Bigl(x_k + \frac{a}{x_k}\Bigr). \tag{3}$$

Here we record some observations on this algorithm, and relate them to the approximation (1).

To start, suppose we are given $a > 0$ and an approximation $x_k > 0$ to $\sqrt{a}$:

$$x_k - \sqrt{a} = \delta_k. \tag{4}$$

We don't know $\sqrt{a}$ and we don't know δ_k, though of course we'd like to have an estimate on δ_k. To proceed, square (4) to obtain $x_k^2 + a - 2x_k\sqrt{a} = \delta_k^2$, hence

$$\sqrt{a} = x_{k+1} - \frac{\delta_k^2}{2x_k}, \quad x_{k+1} = \frac{x_k^2 + a}{2x_k} = \text{ right side of (3)}. \tag{5}$$

Consequently

$$x_{k+1} - \sqrt{a} = \delta_{k+1}, \quad \delta_{k+1} = \frac{\delta_k^2}{2x_k}. \tag{6}$$

One immediate consequence of (6) is that, whenever $\delta_k \neq 0$, δ_{k+1} is positive, hence $x_{k+1} > \sqrt{a}$. Consequently, if we start with an initial guess $x_0 > 0$ to $\sqrt{a}$, we get $x_k > \sqrt{a}$ for all $k \geq 1$. Thus (if $\delta_0 \neq 0$), for all $k \geq 1$,

$$\frac{a}{x_k} < \sqrt{a} < x_{k+1} < x_k. \tag{7}$$

In particular, we have an estimate on δ_k:

$$\delta_k < x_k - \frac{a}{x_k} = \frac{x_k^2 - a}{x_k}. \tag{8}$$

Note also that a/x_k is a bit closer to $\sqrt{a}$ than is x_k. We have

$$\sqrt{a} - \frac{a}{x_k} < \frac{x_k^2 - a}{2x_k} < x_k - \sqrt{a}. \tag{9}$$

On the other hand, once this algorithm really kicks in, δ_{k+1} is much smaller than δ_k, and the differences $x_k - \sqrt{a}$ and $\sqrt{a} - a/x_k$ are relatively close to each other.

To relate these observations to the approximation s to $\sqrt{2}$ given by (1), note that comparison with (2) (or squaring both sides of (1)) gives

$$s < \sqrt{2}. \tag{10}$$

Consequently the quantity s is not going to appear on a list (x_k) of approximations to $\sqrt{2}$ obtained by the algorithm (3). This leads us to wonder if there is some variant of the algorithm (3) that can lead to the approximation s in (1).

Taking a clue from (9), we might consider the possibility that

$$s = \frac{2}{x_k}, \quad x_k = 1.41421415, \tag{11}$$

with x_k obtained from some initial guess x_0 via the algorithm (3). As (9) implies, and as one can verify by comparing (1) and (11) with (2), this number s is a (very slightly) better approximation to $\sqrt{2}$ than x_k. The problem with this scenario is that it takes a bit of computation to pass from x_k to $s = 2/x_k$, and then just a tiny bit more to pass to $x_{k+1} = (x_k + s)/2$, an approximation with a much smaller error. Somehow, it is hard to imagine someone taking this route without going one small step further.

We now examine another iterative approximation to square roots. In fact, this method readily applies to nth roots. So take an integer $n \geq 2$, and consider $a > 0$. We desire to approximate $a^{1/n}$, starting from an initial guess y_0. This time we use the following iteration. If $y_k > 0$ is an approximation to $a^{1/n}$, set

$$y_k^n = a(1 + \beta_k), \tag{12}$$

and define the next approximant by

$$y_{k+1} = y_k \Bigl(1 + \frac{\beta_k}{n}\Bigr)^{-1}. \tag{13}$$

For example, approximating $2^{1/3}$ with initial guess $y_0 = 1$ yields

$$\begin{aligned} y_1 &= 1.2 \\ y_2 &= 1.25698\cdots \\ y_3 &= 1.2599141\cdots \\ y_4 &= 1.259921049857\cdots \\ y_5 &= 1.259921049894873, \end{aligned} \tag{14}$$

with $|y_5^3 - 2| < 10^{-15}$.

For $n = 2$, the iteration (12)–(13) gives

$$y_{k+1} = \frac{2ay_k}{a + y_k^2} = \frac{a}{x_{k+1}}, \tag{15}$$

where x_{k+1} is obtained from $x_k = y_k$ by (3), i.e.,

$$x_{k+1} = \frac{1}{2}\left(y_k + \frac{a}{y_k}\right). \tag{16}$$

It follows that $y_{k+1} < \sqrt{a}$ for $k \geq 0$, hence $y_k < \sqrt{a}$ for $k \geq 1$, and, in counterpoint to (7),

$$y_k < y_{k+1} < \sqrt{a} < \frac{2}{y_k}. \tag{17}$$

This therefore provides a route to the approximation (1). Is this the route taken in 1800 BC? Maybe. One contraindication is that (12)–(13) applies equally well to cube roots, fourth roots, fifth roots, etc., but it does not seem that the ancients took advantage of this. (See however remarks about Archimedes below.)

Let's just try to emulate what could have been done in 1800 BC. These days, knowing that $14^2 = 196$, we who use decimals would be inclined to start with $x_0 = 1.4$. Back then they used sexagesimals, e.g., $x_0 = 1 + \frac{4}{10} = 1 + \frac{24}{60}$. But it is easy to see that $x_0 = 1 + \frac{25}{60}$ is a better approximation to $\sqrt{2}$, so we will start there. Then using the iterations (3) and (15) yields the following:

$$\begin{aligned} x_1 &= 1.414215686274510, & y_1 &= 1.414211438474870, \\ x_2 &= 1.414213562374690, & y_2 &= 1.414213562371500, \\ x_3 &= 1.414213562373095, & y_3 &= 1.414213562373095. \end{aligned} \tag{18}$$

We see that y_1 is somewhat close to s in (1), but not at all spot on. The error $\sqrt{2} - y_1$ is $\approx 21 \times 10^{-7}$, somewhat larger than the error for s. Note that the errors in x_2 and y_2 are $\approx 16 \times 10^{-13}$.

We pass from efforts to see where (1) comes from and make some further comments about the evaluation of square roots. Though the method (3) is called the Babylonian method and said to go back many centuries BC, so many records from those times have been lost that the earliest explicit record of use of (3) seems to be in work of Heron (about 100 AD). To be sure, there are numerous records of square roots predating this. We mention just one exemplary case, work of Archimedes.

To approximate π, Archimedes evaluated the perimeters of regular n-gons P_n, inscribed in a unit circle, for $n = 3 \cdot 2^k,\ 2 \le k \le 5$. In modern terminology,

$$\ell(P_n) = n|e^{2\pi i/n} - 1|, \tag{19}$$

where, for $|\theta| < \pi/2$,

$$|e^{2i\theta} - 1| = |e^{i\theta} - e^{-i\theta}| = 2\sin\theta, \tag{20}$$

so

$$\frac{1}{2}\ell(P_n) = n\sin\frac{\pi}{n}, \tag{21}$$

which tends to π as $n \to \infty$. Of course, $\ell(P_6) = 6$. One proceeds to compare $\ell(P_n)$ and $\ell(P_{2n})$, which comes down to comparing $\sin\theta$ and $\sin\theta/2$.

To get this, we can take the classical formula for angle bisection, written as

$$e^{i\theta/2} = \frac{e^{i\theta} + 1}{|e^{i\theta} + 1|}. \tag{22}$$

and compare imaginary parts, to obtain, for $|\theta| < \pi$,

$$\sin\frac{\theta}{2} = \frac{\sin\theta}{\sqrt{2 + 2\cos\theta}}, \tag{23}$$

noting that

$$\cos\theta = \sqrt{1 - \sin^2\theta}, \quad \text{for} \quad |\theta| < \frac{\pi}{2}. \tag{24}$$

Repeated application of (23)–(24) leads to the evaluation of (21) for $n = 3 \cdot 2^k$.

Clearly evaluating $\ell(P_{96})$ this way involves calculating a number of square roots. Did Archimedes use the Babylonian method to do this? Maybe. Many records of what Archimedes did are lost. He is said to have continued the approximation of π much further, in his *Book of Circles*, most of which is lost. He is said to have had a way to compute cube roots, and to have prepared a trig table. All lost.

There is work of Ptolemy on a trig table, done centuries later, that survives. This includes (the equivalent of) a table of values of

$$\cos\ell^\circ,\ \sin\ell^\circ, \quad \ell \in \{1, 2, , \dots, 90\}, \tag{25}$$

where $\ell^\circ = \pi\ell/180$ radians. To produce this, one starts with classical evaluations of $\cos\theta$ and $\sin\theta$ for $\theta = \pi/3$, $\pi/4$, and $\pi/5$, involving various square roots. Then bringing in algebraic identities, such as

$$\begin{aligned} e^{\pi i/3}e^{-\pi i/4} &= e^{\pi i/12}, \\ e^{2\pi i/5}e^{-\pi i/3} &= e^{\pi i/15}, \\ e^{\pi i/12}e^{-\pi i/15} &= e^{\pi i/60}, \end{aligned} \tag{26}$$

one can obtain evaluations of $\cos\ell^\circ$ and $\sin\ell^\circ$ for ℓ an integral multiple of 3. In particular,

$$\begin{aligned} \sin 3^\circ &= 0.052375956242\cdots, \\ \cos 3^\circ &= 0.998629534754\cdots. \end{aligned} \tag{27}$$

As it turns out, one cannot get to $\cos 1^\circ + i\sin 1^\circ = e^{\pi i/180}$ simply by taking square roots. Ptolemy shows that

$$\frac{2}{3}\sin\frac{3}{2}^\circ < \sin 1^\circ < \frac{4}{3}\sin\frac{3}{4}^\circ \tag{28}$$

and evaluates $\sin(3/2)^\circ$ and $\sin(3/4)^\circ$ from (27) and the half angle formula (23), obtaining a not particularly accurate approximation to $\sin 1^\circ$.

For another approach to $\sin 1^\circ$, one can write

$$\cos 1^\circ + i\sin 1^\circ = a^{1/3}, \quad a = \cos 3^\circ + i\sin 3^\circ, \tag{29}$$

and use the fact that the method (12)–(13) extends from a computation of $a^{1/n}$ for $a > 0$ to the setting of complex a. In the setting of (29), we have

$$a = 1 + b, \tag{30}$$

with $b \in \mathbb{C}$, $|b| << 1$, and we can use (12)–(13) with initial guess

$$y_0 = 1 + \frac{b}{3}. \tag{31}$$

For more details on this, and further material on making a trig table, one can see Chapter 2, §8 of [T].

References

[K] M. Kim, History, identity, and ownership in mathematics, Notices AMS, 72, #8, Sept. 2025, 846–853.

[T] M. Taylor, The Complex Plane and the Euclidean Plane – Algebraic Approach to Plane Geometry, Preprint, 2025.